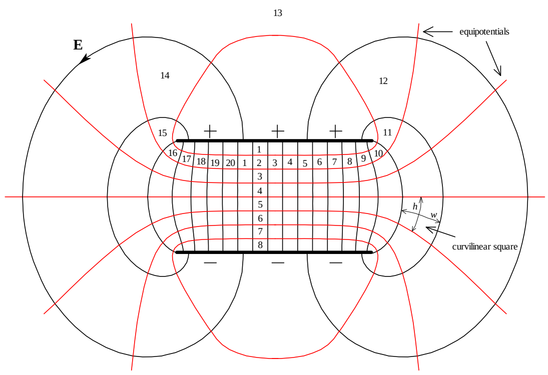
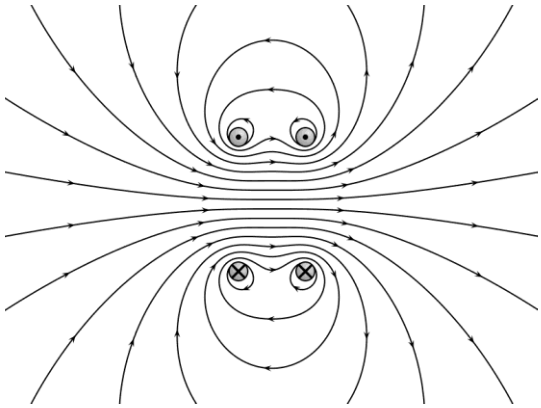
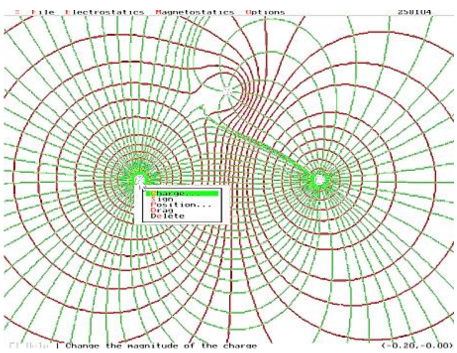
48xxx Fields and Waves

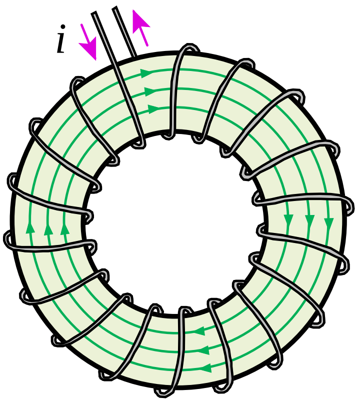
Topic Notes

2019









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Preface

These topic notes comprise part of the learning material for *48xxx Fields and Waves*. They are not a complete set of notes. Extra material and examples may also be presented in the face-to-face activities.

Using the electronic version of these notes

These notes are hyperlinked. All green text is a link to somewhere else within this document. For example, the contents page links to the appropriate page in the text, and the page numbers in the header on each page link back to the contents page. There are also some internal linked words that take you to the relevant text.

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Introduction

Electric circuit theory and electromagnetic theory are the two fundamental theories upon which all branches of electrical engineering are built. Many branches of electrical engineering, such as power, electric machines, high speed digital circuits, electronics and communications are based on electromagnetic theory. Electromagnetic theory is also valuable to students specializing in other branches of the physical sciences because of the applied mathematics and physics involved.

At the undergraduate level, it is important to impose a particularly disciplined approach to electromagnetic theory that emphasises the conceptual meanings, rather than the mathematical rigour, in the study of fields.

Electromagnetic theory has traditionally appeared to practically-oriented prospective engineers as an academic luxury. Nevertheless, field theory studies have stood the test of time as a portion of virtually all electrical engineering degree courses, because it forms the basis of nearly all the macroscopic equations and concepts employed in electrical engineering. This subject addresses the following areas, in a manner that may help students appreciate that the 'luxury' is worthwhile:

a) An elaboration on the foundation that supports much of circuit theory, machine theory and high-frequency behaviour of electrical systems. The weaknesses and strengths of this foundation tend to produce surprises to accepted notions held by thinking students.

b) The use of this foundation to derive many of the common equations already used by students, pointing out their range of applications as defined by their derivation.

c) The consequent development of new notions for energy flow and current patterns that may assist in a more mature appreciation of high-frequency behaviour, interference and capacitive effects, machine and transmission line modelling.

d) Providing a 'window' to look into some special areas of electrical engineering where field concepts must be used explicitly. Optical and metal waveguides, radiating antennae and receivers, and effective resistances of transmission lines are chosen as the most common examples of these areas.

e) Providing better facility in many mathematical techniques necessary for the aims above, with the hope that this facility permeates to other subjects (particularly in Circuits, Signals and Control strands) using similar techniques. For effective mathematical teaching to engineers, the stress should continually be placed on the physical concepts behind the symbols and operators.

f) The application of some of the electrical system models to other areas of engineering, particularly mechanical, and to familiar everyday systems, to illustrate the wider basis of the mathematical tools.

In human terms, the subject aims to create engineers of a more 'professional' character than would otherwise be the case. Their conceptual knowledge of electromagnetic theory will provide a broader horizon for problem-solving thinking patterns, perhaps making specific techniques a little more enjoyable and meaningful. They may have more confidence (and a little more competence) in assessing an unfamiliar problem, or a problem that apparently contradicts normal circuit behaviour. They may recognise where broader-based field notions must be applied, even if in many cases their experience is not sufficient to develop solutions.

In essence, electromagnetic field theory can be one sma1l tool applied to developing a flexible, analytic and critical mind in a technologically-changing world that needs all the flexible minds it can muster.

Supporting References

There are a vast variety of teaching methods employed in electromagnetic field theory, reflected in the character of texts available. A subset of the volumes that are roughly similar in nature to this subject are listed below:

1. Plonus, M.: *Applied Electromagnetics*, McGraw-Hill, 1978.

   An excellent book that provides a wealth of examples and applications, and doesn't compromise on theoretical and mathematical rigor.

2. Ramo, S., Whinnery, J.R. and Duzer, J.V.: *Fields and Wave in Communications Electronics*, 2nd *Ed*., John Wiley & Sons, 1984.

   An intermediate-level text which has a strong focus on communications (waveguides, microwave networks, antennas and optics).

3. Magid, L.: *Electromagnetic Fields, Energy and Waves*, Robert Krieger, 1982.

   This reference is the best overall choice for its exhaustive attempts to provide physical meaning by way of discussion and example.

4. Solymar, L.: *Lectures on Electromagnetic Theory*, Oxford 1976.

   The concise descriptions of selected parts of electromagnetism make this volume good introductory 'light reading' to provide a feel for the subject.

5. Rao, N.: *Elements of Engineering Electromagnetics*, Prentice Hal1, l977.

   The notation used by Rao (and by Magid) appears simpler to students than some other alternatives, and the book covers the later parts of the subject with some distinction.

6. Johnk, C.: *Engineering Electromagnetic Fields and Waves*, Wiley, 1975.

   This book is reasonably comprehensive, and particular topics are easy to locate. The treatment is fairly lucid, but suffers from a notation that is not universally clear to new students.

7. Skitek, G. and Marshall, S.: *Electromagnetic Concepts and Applications*, Prentice-Hall, 1982.

   A clear treatment with worked examples, this book should serve well as a support for study with the exception of some gaps in its coverage.

8. Cheng, D.K.: *Field and Wave Electromagnetics*, Addison-Wiley, 1983.

   Presents the material with lucidity, unity and smooth logical flow of ideas. Many worked out examples are included to emphasize fundamental concepts.

9. Griffith, D.: *Introduction to Electrodynamics*, 4th *Ed*., Pearson, 2013.

   A 'standard' physics textbook that presents the concepts in a less formal tone, but it lacks engineering context and applications.

10. Pipes, L. and Harvill, L.: *Applied Mathematics for Engineers and Scientists*, 3rd *Ed*., McGraw-Hill, 1971.

    A book that covers all the fundamental mathematical techniques used by engineers, although the terminology is inconsistent and a little dated.

Synopsis

Topics 2 and 4 lay the basis of the macroscopic field concept. The electrostatic field is considered due to an isotropic emission of 'fluid' from every source point. The motion of such sources causes relativistic distortions of spatial dimensions, and in turn distorts the density of field lines. This distortion of the electric field is known as the magnetic field.

Topics 3 and 5 develop these concepts using vector calculus, to provide a summary of the basic equations encompassing electrostatics and magnetostatics.

Topics 6 and 7 develop methods of solving static problems, particularly involving Laplace's equation. Many non-electrical systems also satisfy Laplace's equation, and these are mentioned in Topic 7. A variety of mechanical, hydraulic and thermal systems are modelled by Poisson's equation, Laplace's equation and the diffusion equation.

Topic 8 extends the methods of solution of differential equations, and discusses the particularly important geometry of the twin-cable, two-line-charges or long-dipole problem.

The notion of energy residing in a field, discussed in Topic 9 extends the audacity of fluid concepts (of Topic 2) to their logical limit, and the remaining topics develop the notions to time-varying cases.

1 The Concept of a Field

Contents

Introduction

The study of fields is concerned with predicting interactions between things that are not apparently touching . There are a wide variety of events that cannot be explained by mechanical contact forces, and the problem of adequately visualising such 'actions at a distance' has been with us for several hundred years.

Prehistoric man could presumably understand that a spear or boomerang would affect an animal if contact could be made, but may have puzzled over the effect that a fire could have on his body when placed several feet away. 'Mechanical' models were eventually used to explain thermal effects in terms of a heat fluid (‘therma') that seeped from 'hot' bodies. (It is only relatively recently that such a model has lost favour. Perhaps you can explain why it is considered an unrealistic model?).

The oral communication practiced between prehistoric people is another action-at-a-distance phenomena. Recorded propositions to explain the transport of sound information, or sound energy, included of course another fluid mechanical explanation; a wave motion model of stress-strain effects in the medium is a relatively recent innovation.

If we study matter itself much more closely, using an atomic or nuclear 'microscope', we find more actions-at-a-distance. Entities within the nucleus (nucleons) include many with positive charge as well as neutral ones. How does the nucleus stay together? Current models have two nucleons held together by a third particle that continually links them. The mutual influence of attraction is transmitted from one nucleon to the other by this 'muon'. Chemical compounds often stay together because the attractive interaction between the two elements is carried by a third, shared particle known as a valence electron.

The following figure summarises the distance force types based on contemporary quantum models, each considered to convey their influence via a third connecting particle.

FORCE	RANGE	STRENGTH AT 10^{-15} METERS IN COMPARISON WITH STRONG FORCE	CARRIER	MASS AT REST (GeV)	SPIN	ELECTRIC CHARGE	REMARKS
GRAVITY	INFINITE	10^{-38}	GRAVITON	0	2	0	CONJECTURED
ELECTRO-MAGNETISM	INFINITE	10^{-2}	PHOTON	0	1	0	OBSERVED DIRECTLY
WEAK	LESS THAN 10^{-18} METERS	10^{-13}	WEAK SECTOR BOSONS: W+ W- Z0	 81 81 93	 1 1 1	 +1 -1 0	OBSERVED DIRECTLY OBSERVED DIRECTLY OBSERVED DIRECTLY
STRONG	LESS THAN 10^{-15} METERS	1	GLUONS	0	1	0	PERMANENTLY CONFINED

Table 1.1 – The 4 Fundamental Forces and Their Conveying Particles[1]

We are concerned with a particular subset of these forces, namely those where the interactions between neighbouring points decrease with distance according to a square law. There are three traditional areas of observation where the square law appears quite clearly; gravitational attraction between masses, electrostatic and magnetostatic influences between charges and currents, and thermal radiation. We will presently find that the model we use to explain these will naturally extend to a variety of other uses, particular in electrodynamics, (electro) magnetic machines, power transmission, electrical communications, high-frequency interference in circuits and high-voltage corona.

[1] Refer to the article *The Higgs Boson*, by Martinus Veltman in *Scientific American*, Nov 1986. The Higgs force is a fifth one, which is mediated by the Higgs boson. The Higgs boson was discovered in 2012 by the LHC at CERN.

In addition, the model will provide valuable insight into familiar equations used in electrical engineering that are not normally considered as field-related. Given a certain faith in the model, we can answer some basic questions about electrical behaviour that are normally 'glossed over' or ignored. For example, how is it that a resistor can be influenced (getting hot) by an AC signal generator some distance away. After all, there is no nett transport of electrons from one to the other.

In summary, we need to develop a consistent model for events where no mechanical explanation is apparent. The model should give the right answers in all macroscopic interactions (or at least those obeying inverse-square laws) but also should alleviate our psychological hang-ups regarding action-at-a-distance. The model we use can be vastly simpler than the particle model used in quantum electrodynamics, since we will never attempt to explain sub-atomic effects.

1.1 Experimental Landmarks

Let us quickly review some basic observations that beg an explanation:

1.1.1 Newton's Law of Gravitation (1665)

In 1665, Newton proposes a gravitational law for attraction at a distance between point masses:

Newton's Law of Gravitation

$$\mathbf{F}_{12} = \frac{Gm_1m_2}{r_{12}^2}\hat{\mathbf{r}}_{12} \tag{1.1}$$

$\mathbf{F}_{12}$ is the force experienced by point mass m_1 due to the influence of m_2, at a separation of r_{12} from m_1.

The quantities expressed in this law are shown below:

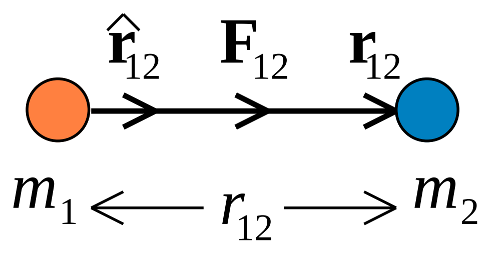


Figure 1.1 – Vectors in Newton's Law of Gravitation

The unit vector $\hat{\mathbf{r}}_{12}$, which points from m_1 to the origin of the force, i.e. from measuring point to source, is defined by:

$$\hat{\mathbf{r}}_{12} = \frac{\mathbf{r}_{12}}{|\mathbf{r}_{12}|} \tag{1.2}$$

Later knowledge of the Calculus allowed this law to be proved for spheres of uniform density, as well as being applied (by summation) to bodies of any finite shape and density distribution.

1.1.2 Priestley's Electrostatic Investigations (1767)

In 1767 Priestley deduced a similar $1/r^2$ distance dependence for separated electrostatic charges. The literature describes European experiments with suspended point charges enclosed by hollow charged conductor spheres. No force was experienced on such an enclosed charge irrespective of its position within the sphere:

Priestley's electrostatic experiment

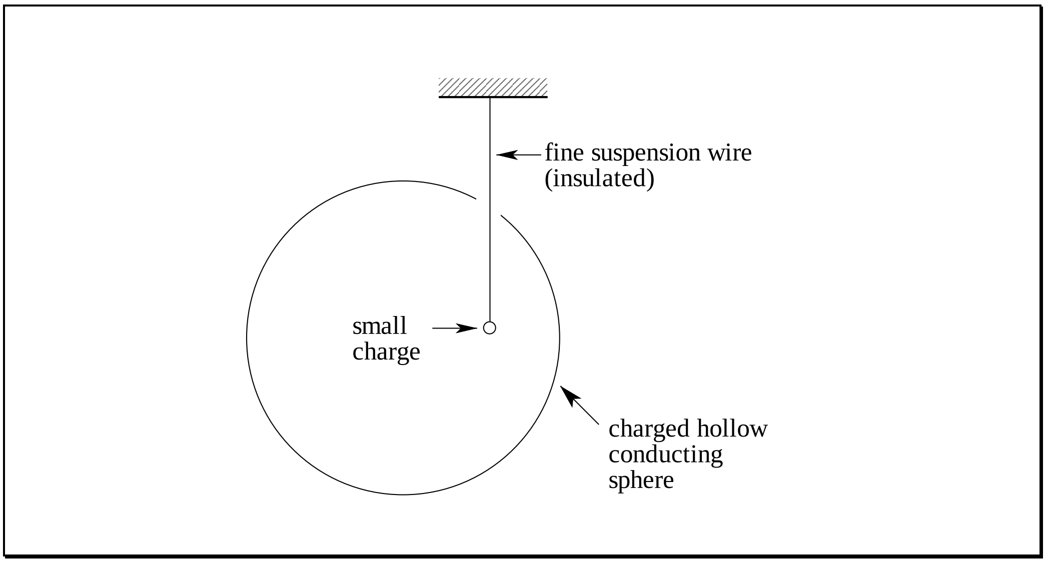


Figure 1.2 - The experimental arrangement that led Priestley to conclude an inverse square law (the charge experienced no motion)

This fact led Priestley to conclude that:

$$F \propto q/r^2 \tag{1.3}$$

where q is the magnitude of the source charge, and r the separation of this charge from the suspended test charge. *Why?*

This $1/r^2$ electrostatic law was verified by direct experiments of Coulomb, although the first reasonably precise measurements were performed by Cavendish, in which:

$$\mathbf{F}_{12} \propto \frac{q_1 q_2}{r_{12}^n} \hat{\mathbf{r}}_{12} \quad \text{where } n = 2 \pm 0.05 \tag{1.4}$$

For the SI system of units we define the proportionality constant such that we get the "modern" form of Coulomb's Law:

$$\mathbf{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0 R^2}\hat{\mathbf{R}} \quad \text{N} \tag{1.5}$$

Coulomb's Law

The quantities expressed in this law are shown below:

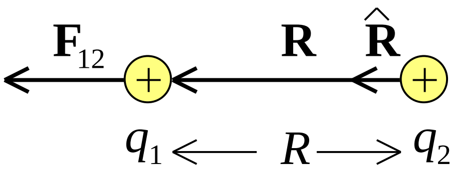


Figure 1.3 – Illustration of terms involved in Coulomb's Law

The subscript '12' on the force means this is the force on charge q_1 due to charge q_2. Therefore, the unit vector $\hat{\mathbf{R}}$ points in a direction that goes from a positive q_2 to positive q_1 (or from a negative q_2 to negative q_1), i.e. the unit vector aligns with the direction of repulsion. Attractive forces will therefore be negative, and are taken care of by the negative sign of either q_1 or q_2.

Here we should clarify some notation. The *vector* **R** points from the source of the force (q_2) to the point where the force is felt (q_1). This will *always* be the case throughout these notes. **R** points from the source to the effect. The magnitude of the vector **R** is just R. The unit vector $\hat{\mathbf{R}}$ has the same properties as **R** except its magnitude is one.

Mathematical notation

1.1.3 Magnetostatics (early 20th century)

A similar action-at-distance law has been found useful in describing the interaction of magnetic "poles":

$$\mathbf{F}_{12} \propto \frac{q_{m1} q_{m2}}{r^2} \hat{\mathbf{r}} \tag{1.6}$$

where q_m represents a magnetic moment.

Each of the above phenomena have prompted an inverse square law between neighbouring points. Of course, the use of calculus enables these point-interactions to be used for other shapes. However, the empirically-based formulae do not provide us with a *picture* of the interaction phenomena. Without such a model, the equations remain as simply tools for prediction of events within their separate domains. A conceptual model allows possibilities for extension to other seemingly-unrelated effects and the predicting of new areas for discovery, or for clearer understanding of related events. It is an avenue worth pursuing.

1.2 Models for Action-at-a-Distance

Since no mechanical contact is apparent in each of these three forms of interaction, a multitude of models has been proposed in order to visualise the mechanisms of action-at-a-distance. The main hypotheses that have been in vogue from time to time are summarised below.

1.2.1 Stress-Through-the-Medium

Stress waves may be transmitted through an intermediary medium by the source particle and picked up by the other particle as a force.

This concept was particularly in vogue in the nineteenth century, when it was generally considered that even outer space was not empty. A substance of zero mass-density pervaded all space, and carried stress-strain waves from sources such as electrical charges, masses or light-emitters. This substance, known as the *ether*, provided a means for effects to be transmitted across space, and these effects could be quantified using methods already developed for mechanical elasticity studies.

In 1881 Michelson devised an experiment to measure the velocity of the ether with respect to the Earth's surface. He, assisted later by Morley, found that waves of light travelled at exactly the same speed irrespective of the direction they propagated in the moving ether. If the ether was supporting the light waves, their velocity would surely be relative to this medium. This was not so. The demise of the ether model can be dated from these measurements, constituting one of the most important negative results in experimental physics.

A conceptual model stands to a large extent on its simplicity, and its straight-forward application. The stress-through-the-medium ether model had also become quite complicated. The ether was attributed with several strange properties, in addition to its zero mass, in order to explain experimental observations in wave optics.

Its lack of simplicity, combined with Michelson and Morley's demonstration of its lack of reality, put an end to the ether concept.

1.2.2 Particle Emission

The interaction of bodies at a distance may be via the exchange of particles. Each body may emit particles which by "collision" transfer a force to the other. This is indeed the current thinking to describe several forms of particle interaction, in chemical bonding and nuclear physics.

For gravitational or electromagnetic interactions, the particle-emission model suffers on both of the main requirements of a model. Firstly, there has been no associated particle yet found that is associated with the gravitational field. The electromagnetic force can be described by the exchange of virtual photons, but the model is not simple to use (nor mentally-satisfying); its complications in explaining attraction as well as repulsion and even simple dielectric properties can be imagined.

1.2.3 Fluid Emission

Interaction at a distance may be via a fluid, independent of any medium. Each body may emit the fluid, in order to transmit forces to neighbouring bodies.

1.3 The Fluid Emission Model

The idea of an all pervading 'ether' must have seemed quite plausible before Michelson and Morley, despite being far-fatched when viewed by hindsight. The notion of fluid emission seems just as absurd, in an experimental sense. However, to explain all the macroscopic properties observed in the gravitational and electromagnetic world, the fluid needs just three simple properties (the combination of these properties can be observed):

Postulate 1

All force-producing points (or particles) emit fluid ISOTROPICALLY. (1.7)

The points may be the things we commonly term 'charges' or 'masses'. The 'fluid' differs from usual fluids according to this postulate, because we would expect most other fluids to show anisotropy when the source moves. For example, water would perhaps emit more volume backwards than forwards when a tap is moved at high speed.

Accordingly, we normally call this special fluid a special name, *flux*, symbolised by Ψ. It is a scalar, just like the volume (and flow rate) of any fluid.

Postulate 2

The density of flux Ψ (per unit cross-sectional area) found at any point in space is proportional to the force transmitted to that point. (1.8)

That is, Ψ is a scalar, indicating the volume of 'fluid' emitted, but the density of Ψ must be a vector.

The density **D**, in units of flux per unit area, tells us the magnitude and direction of the force potentially experienced (by another charge or mass) at each point.

Postulate 3

The total flux emitted from a point source is not changed if the source is at rest or in motion. (1.9)

We will find these three simple notions adequate to explain all our (macroscopic) Electrical Engineering Laws, and many more phenomena in engineering. We cannot explain observations on an atomic or nuclear scale, where quantum electrodynamic field theory must be used. The wave-particle duality employed in the latter theory can be used to explain the macroscopic observations of engineering as well, but it is exceedingly cumbersome. Given the hindsight of quantum mechanics, it is possible to gain a better picture of why the above flux picture works, but in practice it is a luxury not worth our time pursuing. Suffice it to say that 'flux' has never been observed as a physical fluid (apart from its three postulated behaviours above) and no doubt never will be.

Despite these comments, the model of flux emission amply satisfies the second property of a good model; it is exceedingly simple. For this reason alone, it has stood the test of time as a very useful mathematical model in those disciplines where a fundamental inverse square-law has been observed. Consider the three postulates again, and take them as an article of faith. You will not need to call on your imagination again, given that these postulates are accepted.

Finally, we finish this outline of a model for action-at-a-distance by a definition. Every particle, or point source, does not produce the same effect on its neighbours. There must be a quantity defined to express this variable influence:

Definition: The 'strength' of a source of flux is the total flux emitted from the source. The magnitude of this 'source strength' is called the *charge* (if the flux transmits electric forces) or the *mass* (for gravitational interactions).

The total Ψ being emitted from a source	=	Charge q at the source	(1.10)

Total flux = charge at the source

Since Ψ is a scalar, so is charge q. In SI units q is in Coulombs. Accordingly, from this definition, Ψ is also in Coulombs.

These 'everyday' quantities of charge and mass have never been observed. The presence of flux Ψ can at least be seen by its influence (transmitted forces), and can therefore be indirectly observed. If you are happy with 'charge' and 'mass', then is it not reasonable to expect that we henceforth take for granted that 'flux' exists?

There are three basic equations used in electrostatics. They can be readily seen as a consequence of the properties of flux (as listed above). Gauss' Law is simply an expression of the meaning of charge. Coulomb's Law must follow for a flux emission that is isotropic. The final equation, Poisson's equation, is a statement that follows from Coulomb's Law and a minor facility with vector calculus. Firstly, we consider Gauss' Law.

1.4 Gauss' Law

Consider a situation where we have a source of flux (a 'charge', or set of charges) in some volume. We wish to know how much flux is being emitted. We may not be able to see the flux coming out, but we can detect it by its property as defined by Postulate 2. We can measure its influence through the force that it transmits.

So we put a force-measuring instrument, say a 'test' charge attached to a spring balance that responds to the repulsive force, at points just outside the volume. Then, at each point:

$$\text{The measured force } \propto \mathbf{D} \tag{1.11}$$

Given that our measuring spring balance subtends a small area dA, a small amount of fluid will be intercepted by the area. The fluid detected will increase as the area of the instrument's face increases, but will also change with the orientation of the face; see the figure below:

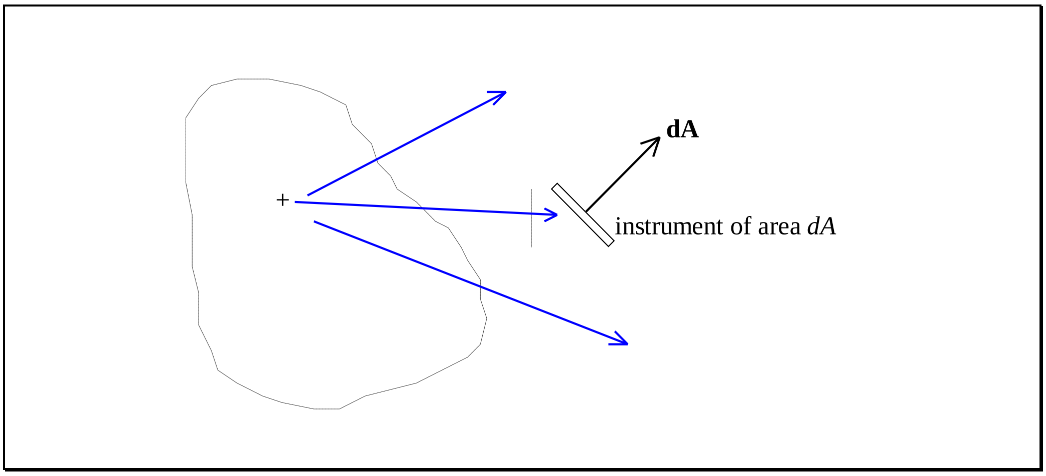


Figure 1.4 – Measuring force due to a flux

The maximum amount of flux will be detected when the face is a plane directly across the flow; that is when **dA** is parallel to **D**. In general, only the component of **dA** that is parallel to **D** will be effective in intersecting flux lines. Thus the instrument accounts for a flux of:

$$d\Psi = \mathbf{D} \cdot \mathbf{dA} \tag{1.12}$$

To detect all the flux emitted from the volume, we need to move the detector around the periphery of the volume, to gather the effect of the other flow lines. We must put the detector at points close enough to one another so as not to miss any of the flow, and the positions must extend around the full periphery, without any gaps. Then the sum of all the measured fluxes, over the completely enclosing surface periphery is:

$$\oint \mathbf{D} \cdot \mathbf{dA} = \text{total } \Psi \tag{1.13}$$

By definition (1.10), this total flux emitted is equal to the total source strength within the volume:

$$\oint \mathbf{D} \cdot \mathbf{dA} = q_{\text{enclosed}} \tag{1.14}$$

Gauss' Law

This is Gauss' Law.

Note that if negative charge exists in the volume, there will be negative source strength at such points. That is, the flux emission will be a negative flow rate, and the points then represent 'sinks' of flux. Flux is absorbed in such cases. It could be, then, that fluid may be flowing into the volume through some parts of the enclosing surface, and the detecting spring balance will in fact be stretched rather than compressed. Influences from sinks of fluid will then appear in the integration sum of Gauss' Law as negative terms. The full summation thus gives a nett outward flux.

$$\begin{array}{c}\text{Nett outward flux}\\ \text{from a volume}\end{array} = \oint \mathbf{D} \cdot \mathbf{dA} = \begin{array}{c}\text{Nett source}\\ \text{strength in the}\\ \text{volume}\end{array} \tag{1.15}$$

This notion, called Gauss' Law by Electrical Engineers, is true in many other fields. For other cases, read **D** as the flux density of interest. For example, the magnetic flux emerging from a volume enclosed by A is:

$$\phi = \oint \mathbf{B} \cdot \mathbf{dA} \quad \text{Weber} \tag{1.16}$$

where **B** is the flux density in Webers/m^2 or Teslas. In fluid mechanics, similar formulae become useful.

1.5 Coulomb's Law

Direct measurement of point charge interaction is extremely difficult (cf. the error estimate in Cavendish's experiment), and even more so for point mass interaction.

However, just as the fluid emission postulates lead simply and directly to Gauss' Law, a concise and general argument can be used to predict a (Coulomb) Law. In the accompanying figure:

Lines of flux streaming from a source

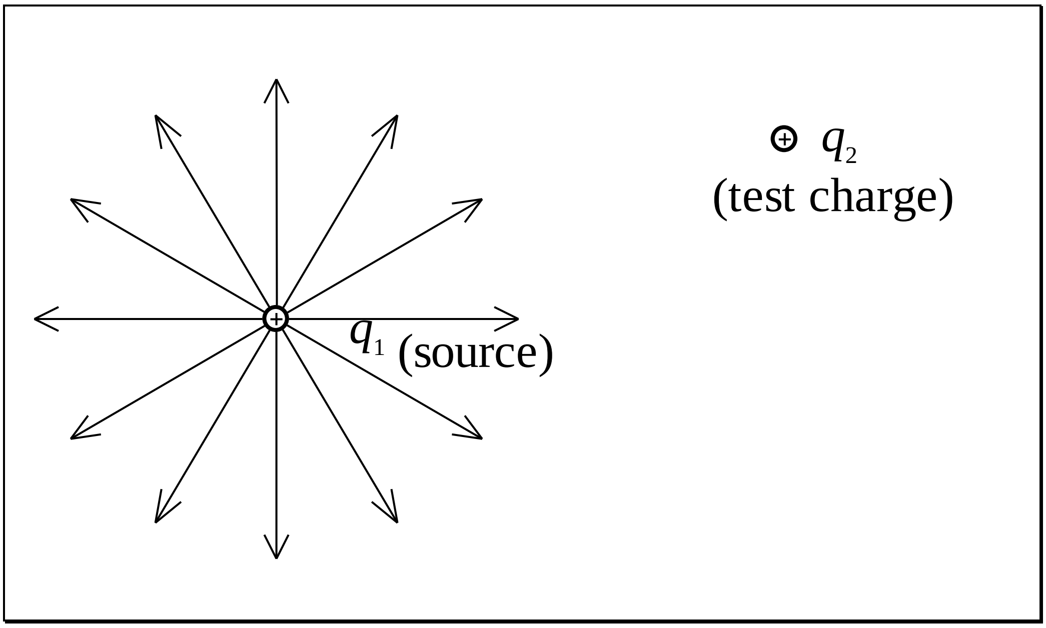


Figure 1.5 – Flux streaming from a point charge

(i) **D**, Ψ, have isotropic distributions from the point source q_1 as represented by the (arrowed) flux lines.

(ii) Flux density **D** at any point is proportional to the force transmitted to that point.

(iii) The force at any point is proportional to the source strength responsible for producing **D**.

Hence the force on q_2 due to q_1 has the following properties:

- $F_{21} \propto D_2$ (i.e. density at 2 due to 1, from (ii) above), and
- $F_{21} \propto q_1$ (from (iii))

Thus, $F_{21} \propto D_2 q_1$. Now using (i) above, the first postulate of the fluid model, we know that **D** is a purely radial vector. In a three-dimensional universe $D \propto 1/r^2$. Check this for yourself by simple geometry. Combining these results together yields:

$$F_{21} = \frac{k_2 q_1}{r^2} \tag{1.17}$$

where k_2 is a proportionality constant that is independent of distance *r* between the source q_1 and test charge q_2.

We could equally well re-draw the figure to envisage q_2 as a source and q_1 the charge experiencing q_2's flux. In this case:

$$F_{12} = \frac{k_1 q_2}{r^2} \tag{1.18}$$

describes the force on q_1, where k_1 is again independent of *r*.

The value of the constant is by no means arbitrary. If we are to satisfy Newton's Third Law of classical mechanics (action = reaction), then $F_{12} = F_{21}$ and:

$$\frac{k_2 q_1}{r^2} = \frac{k_1 q_2}{r^2} \tag{1.19}$$

This result must be true for *any* values of q_1 and q_2. We have not specified these values, nor restricted them in any way. The values of k_1 and k_2 must therefore follow some functional expression (independent of *r* of course) that *ensures* the result will work for any values of q_1 and q_2. (If $q_1 = q_2$ then the *k*'s can be simply equal and constant, but this is not the general case). The only functional dependence that will satisfy Newton's Third Law has *k* proportional to the test charge. That is:

$$k_2 = Cq_2; \quad k_1 = Cq_1 \tag{1.20}$$

Here *C* is a universal constant independent of both charges *and* their separation. We now have:

$$F_{12} = F_{21} = C\frac{q_1 q_2}{r^2} \tag{1.21}$$

In the SI system of units, we write:

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} \tag{1.22}$$

There is no need for the two subscripts on the force symbol *F*, now that both action and reaction produce the same result. We should also note that Newton's Third Law is really a *vector* statement. Action and reaction cancel each other, being equal in magnitude and opposite in sense. The sum of these *vectors* can only come to zero if they are directed along the line of centres, where no nett cross components can occur. We can now state this directional information as:

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \qquad (1.23)$$

Coulomb's Law

where $\hat{\mathbf{r}}$ is the position vector directed from the source (as origin) to the test charge.

This result, Coulomb's Law, followed directly from:

a) the fluid postulates of Section 1.3, and

b) the classical mechanics embodied in Newton's 3rd Law.

It is therefore not surprising that Coulomb's Law is useless in the atomic and nuclear world described by (non-classical) quantum mechanics. Closer to home, we will shortly find it sadly wanting for *any charges in motion*, a world more familiarly described as magnetic field situations.

1.6 A Recipe for Solving Gauss' Law Problems

The main reason for the presence of Gauss' Law in Field Theory courses lies in its utility for understanding and deriving other formulae. However, there are cases with sufficient geometrical symmetry, as exemplified in the Exercises, in which Gauss' Law can be used directly, and are well worth the practice of solving for two reasons. Firstly, a facility with these geometries makes memorizing of results superfluous and allows other cases (given reasonable symmetry) to be quickly figured out. Secondly, an understanding of the Law itself will be much enhanced with practice in its use. Principally with the latter aim in mind, **it is imperative that each step involved in solving such problems be carefully understood and justified.**

The recipe below is a well-proven method to guide you in considering each step. Practice with this recipe should leave you almost immune to failure with unseen Gaussian problems.

A Recipe for Solutions in Guass' Law (integral form)

(Determining flux densities due to symmetrical charge distributions)

1. Draw a picture of the charge (source) geometry. Sketch in **D** flux lines. You can justify your **D** lines from the first postulate of our fluid model (Section 1.3).

2. Choose a Gaussian surface that:

   (i) *intersects the field point* at which **D** is required. (If this condition is not met, the solution for **D** will not be relevant at this required point).

   (ii) *possesses the same symmetry as the charge geometry*. If a point source, then spherical symmetry; if a line, then cylindrical; if a sheet, then flat faces are required.

   (iii) *is* ***closed***. Gauss' Law is not valid for open surfaces. Hence ends and side faces should be added to the surfaces where necessary. These should be chosen with (iv) in mind.

   (iv) *has each face either parallel or perpendicular to the sketched* **D** *lines*. Sketch in the Gaussian surface.

3. Write Gauss' Law $\oint \mathbf{D} \cdot \mathbf{dA} = q_{\text{enclosed}}$.

4. Evaluate the LHS as follows:

   a) Since integration of a vector is not possible, the **dot product must be reduced to a scalar** first. Hence separate the integral into a sum of integrals over surfaces that have, according to 2. (iv):

      - **dA** parallel to **D** lines: $\mathbf{D} \cdot \mathbf{dA} = DdA$
      - **dA** perpendicular to **D** lines: $\mathbf{D} \cdot \mathbf{dA} = 0$

   b) With the Gaussian surface chosen as in 2 (ii), the value of **D** should be constant over the remaining surface where $\mathbf{D} \cdot \mathbf{dA} = DdA$. If it is not, then a more general method (Laplace's or Poisson's equation) should be chosen for the problem. Justify *why* D is constant in your case.

Hence $\int \mathbf{D} \cdot \mathbf{dA} = \int D dA = D \int dA$ since D is a constant.

Sum the area $\int dA$. Arbitrary dimensions for the Gaussian surface can be chosen, since the final answer will necessarily involve their cancellation. The surface is simply a mathematical figment, and its dimensions cannot influence the physical properties.

5. Evaluate the RHS: Sum the charge enclosed within the arbitrary dimensions of the Gaussian surface. If the charge is not uniform, this may involve an integration.

6. Hence now solve the simple linear equation in D.

7. By reference to your sketch, state mathematically or in words the direction of the *vector* **D**. State the units (Cm^{-2}).

E and **F** can, if required, be now found from the expression for **D**. How?

1.7 Summary

- Electrostatic forces were experimentally found to follow an inverse square law, known as Coulomb's Law:

$$\mathbf{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0 R^2}\hat{\mathbf{R}} \quad \text{N}$$

- To model action-at-a-distance we use a "fluid emission" model, which involves the concept of *flux*.

- The fluid emission model leads directly to Gauss' law:

$$\oint \mathbf{D}\cdot d\mathbf{A} = q_{\text{enclosed}}$$

- Coulomb's Law can be derived from Gauss' Law and Newton's 3$^{\text{rd}}$ Law.

1.8 References

http://www.ep.ph.bham.ac.uk/general/outreach/Priestley/
(Accessed 2019-02-28)

Silvester, P.P.: *Modern Electromagnetic Fields*, Prentice Ha11, 1968.

Exercises

1.

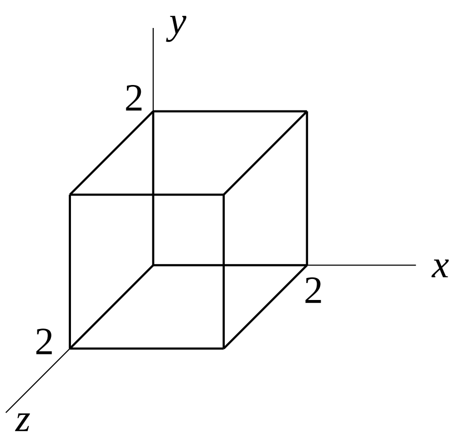
(a) Find the force acting on a small charge q at a distance x along the axis of a circular charged wire ring, of charge density $\lambda\ \text{Cm}^{-1}$ and radius r.

(b) If the distance x is sufficiently large, show that the ring acts as a point charge.

(c) If the ring is extended to a solid disc of radius R and charge density $\sigma\ \text{Cm}^{-2}$, determine the new force acting.

2.

The figure shows a cubic volume, with sides of 2 m length.



Find the charge enclosed within the volume if the flux density **D** is:

$$\mathbf{D} = 2x^2\hat{\mathbf{x}} \quad \text{Cm}^{-2}$$

3.

Find the electric flux density **D**, and the electric field intensity **E** (assuming the medium is air), at a distance r from:

(a) A point charge q.

(b) A line charge of infinite length, with charge density $\lambda\,\text{Cm}^{-1}$ (such as the inner conductor of a coaxial cable).

(c) The axis of an infinitely long, hollow, cylindrical shell of radius R and uniform charge density $\sigma\,\text{Cm}^{-2}$. (Try both $r < R$ and $r > R$).

(d) A uniform large plane surface charge of q Coulombs over the two faces of an area A of conductor.

(e) Above the same conductor plate as (d) when a second plate is also positioned above it at a height $h > r$.

4.

A static electric charge is distributed in the form of a uniform spherical cloud of radius a. Find the electric field intensity at <u>all</u> points. Compare the results with that for a point charge field and explain similarities and differences.

5.

A positive charge q is distributed uniformly over a hollow spherical shell of radius a. A second shell of surface density $\sigma\,\text{Cm}^{-2}$ surrounds it at radius $b > a$.

(a) Find the value of σ that produces zero **E** field at points $r > b$.

(b) What is **E** at points $r < b$ with this value of σ?

6.

A spherical charge distribution is given by:

$$\rho = \begin{cases} K_0\left(1 - \dfrac{r^2}{a^2}\right) & r < a \\ 0 & r > a \end{cases} \quad \text{Cm}^{-3}$$

where K_0 and a are constants.

(a) Sketch the distribution, and a few representative field lines.

(b) Determine the total charge.

(c) Determine the electric field **E** at all points. (Justify each step in your calculations).

Exercises

1.

Verify mathematically:

(a) $\nabla \times \nabla \phi = 0$ for any scalar field.

(b) $\nabla \cdot \nabla \times \mathbf{F} = 0$ for any vector field.

(c) Show that $\nabla^2 \mathbf{F}$ must in general be the sum of 9 terms.

(d) $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$

(e) $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$

(f) $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = -\mathbf{B} \cdot \mathbf{A} \times \mathbf{C}$ (use determinants).

(g) $\nabla \cdot \mathbf{B} \times \mathbf{C} = \mathbf{C} \cdot \nabla \times \mathbf{B} - \mathbf{B} \cdot \nabla \times \mathbf{C}$

(h) $\nabla \cdot [\phi \mathbf{F}] = \phi[\nabla \cdot \mathbf{F}] + \nabla \phi \cdot \mathbf{F}$

(i) $\nabla \{\phi \psi\} = \{\nabla \phi\}\psi + \phi \nabla \psi$

(j) $\nabla \psi \times \mathbf{F} = \nabla \times (\psi \mathbf{F}) - \psi(\nabla \times \mathbf{F})$

(k) $\nabla(1/\mathbf{r}) = -\hat{\mathbf{r}}/r^2$

(l) $\nabla \cdot (\psi \nabla \phi) = \nabla \psi \cdot \nabla \phi + \psi \nabla^2 \phi$

Do (a), (b), (h), (i) and (j) tally with your knowledge of fields, Helmholtz' Theorem and operator behaviour? Why is (f) different from (g)?

2.

Derive the expression for $\nabla \cdot \mathbf{A}$ in cylindrical coordinates.

3.

Derive the expression for $\nabla \cdot \mathbf{A}$ in spherical coordinates.

4.

Derive the expression for $\nabla \times \mathbf{A}$ in cylindrical coordinates.

5.

Derive the expression for $\nabla \times \mathbf{A}$ in spherical coordinates.

6.

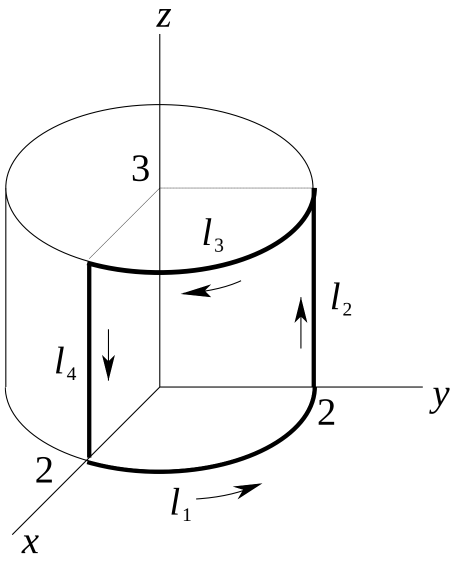
A ϕ -directed electric field in some region is given by:

$$E = Kr^2 z \quad \text{Vm}^{-1}$$

where K is a constant.

(a) Use cylindrical coordinates to find curl **E** at any point. Is **E** conservative?

(b) Evaluate the line integral of $\mathbf{E} \cdot \mathbf{dl}$ taken about a closed path $l = l_1 + l_2 + l_3 + l_4$ on a circular cylinder of radius 2 and height 3 as illustrated:



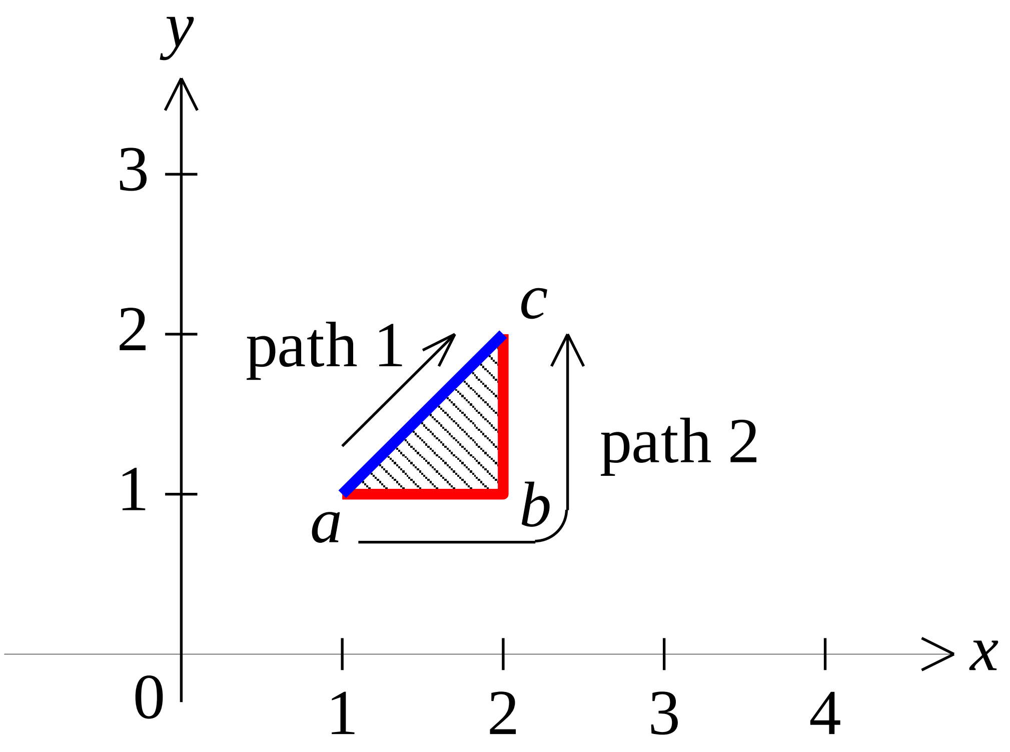
2.62

7.

For the vector:

$$\mathbf{A} = x^2 y^3 \hat{\mathbf{x}} + x^3 y^2 \hat{\mathbf{y}}$$

which exists in the following plane:



Evaluate:

(a) The line integral of $\mathbf{A}$ along path 1 from point a to point c. Repeat for path 2 and compare the results.

(b) The surface integral of $\nabla \times \mathbf{A}$ over the shaded area enclosed by paths 1 and 2.

8.

A scalar line integral of a vector field of the type:

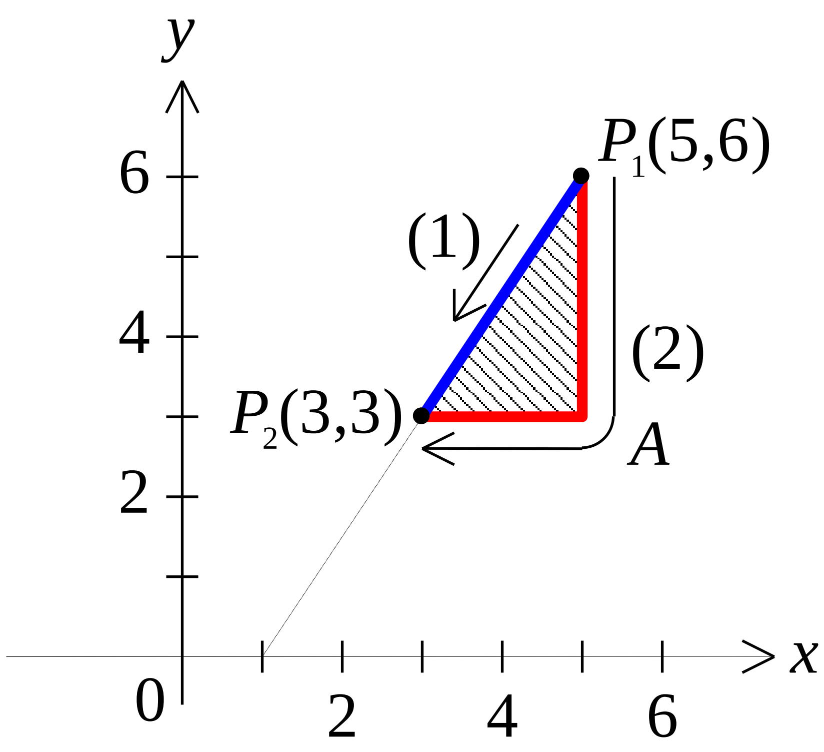
$$\int_{P_1}^{P_2} \mathbf{F} \cdot \mathbf{dl}$$

is of considerable importance in both physics and electromagnetics. (If **F** is a force, the integral is the work done by the force in moving from P_1 to P_2 along a specified path; if **F** is replaced by **E**, the electric field intensity, then the integral represents an electromotive force).

Assume:

$$\mathbf{F} = xy\hat{\mathbf{x}} + \left(3x - y^2\right)\hat{\mathbf{y}}$$

Evaluate the scalar line integral from $P_1(5,6)$ to $P_2(3,3)$ in the figure below along the direct path (1) (P_1P_2), then along path (2) (P_1AP_2):

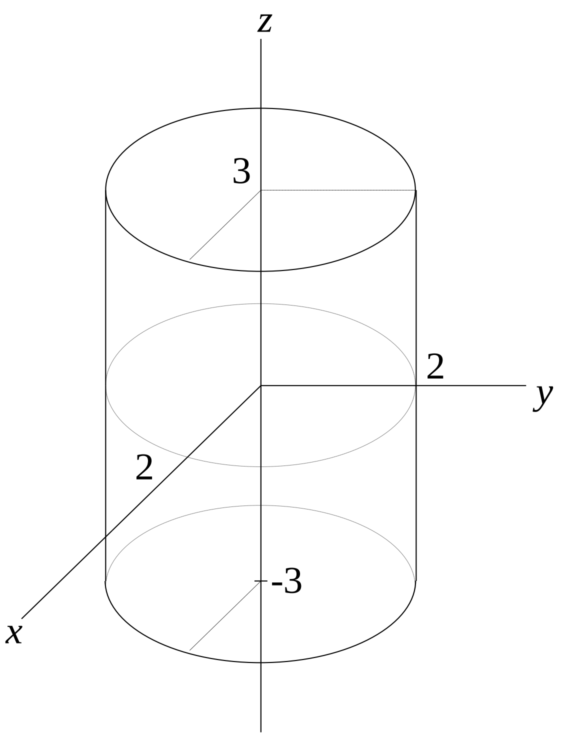


9.

Given:

$$\mathbf{F} = \frac{k_1}{r}\hat{\mathbf{x}} + k_2 z\hat{\mathbf{z}}$$

evaluate the scalar surface integral $\oint \mathbf{F} \cdot \mathbf{dA}$ over the surface of a closed cylinder about the *z*-axis specified by $z = \pm 3$ and $r = 2$, as shown:



10.

Electrostatic fields are derivable as the (negative) gradient of a scalar potential function ϕ:

$$\mathbf{E} = -\nabla\phi$$

(a) Find the electric field $\mathbf{E}$ associated with the following potentials:

$$\phi_1 = V_0(xy+1)$$

$$\phi_2 = V_0 \cosh\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$$

$$\phi_3 = V_0 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$$

where V_0 is a constant voltage.

(b) The electric charge density ρ (in Cm^{-3}) is found to be related to an electric field by:

$$\rho = \varepsilon_0 \nabla \cdot \mathbf{E}$$

where ε_0 is the permittivity of air.

Determine ρ for each of the fields of part (a).

(c) Show that $\nabla \times \mathbf{E} = 0$ for each of the fields.

11.

Given a vector function:

$$\mathbf{F} = (3y - c_1 z)\hat{\mathbf{x}} + (c_2 x - 2z)\hat{\mathbf{y}} - (c_3 y + z)\hat{\mathbf{z}}$$

(a) Determine the constants c_1, c_2 and c_3 if **F** is irrotational.

(b) Determine a scalar potential function *V* corresponding to **F**.

12.

For the vector field:

$$\mathbf{F} = \frac{C}{r^3}\left[(2\sin\theta\sin\phi)\hat{\mathbf{r}} - (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} - (\cos\phi)\hat{\boldsymbol{\varphi}}\right]$$

evaluate:

(a) $\oint \mathbf{F}\cdot d\mathbf{A}$ over a spherical surface of radius *R*, centred at the origin.

(b) Use the physical meaning of divergence; $\oint(\nabla\cdot\mathbf{F})dV = \oint \mathbf{F}\cdot d\mathbf{A}$, to check your answer to part (a) .

(c) Can this field be represented by a scalar potential?

13.

Given the vector field:

$$\mathbf{F} = (r\cos\phi)\hat{\mathbf{r}} - (r\sin\phi)\hat{\boldsymbol{\varphi}}$$

in cylindrical polar coordinates, evaluate:

$$\oint \mathbf{F} \cdot \mathbf{dA}$$

over the quadrant-shaped surface defined by:

$$r = a,\ \phi = 0 \text{ to } \phi = \pi/2,\ z = 0 \text{ to } z = L$$

Note: The differential area vectors in cylindrical polar coordinates can be expressed as:

$$dA_r = rd\phi dz,\ dA_\phi = drdz,\ dA_z = rd\phi dr$$

14.

Helmholtz' theory states that any vector field **F** can be imagined as the sum of two components:

$$\mathbf{F} = \nabla\phi + \nabla \times \mathbf{A}$$

Show that the two components represent an orthogonal system, i.e. if $\mathbf{A} = 0$, show that **F** is irrotational. State one other property of the class of fields for which $\mathbf{A} = 0$. Prove your assertion.

Answers

1.1

(a) $\dfrac{\lambda q x r}{2\varepsilon_0\left(r^2+x^2\right)^{\frac{2}{3}}}\hat{\mathbf{x}}$ (b) $\dfrac{(2\pi r\lambda)q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{x}}=\dfrac{Qq}{4\pi\varepsilon_0 r^2}\hat{\mathbf{x}}$ (c) $\dfrac{\sigma R^2}{4\varepsilon_0 x^2}\hat{\mathbf{x}}$

1.2

32 C

1.3

(a) $\dfrac{q}{4\pi r^2}\hat{\mathbf{r}}$ (b) $\dfrac{\lambda}{2\pi r}\hat{\mathbf{r}}$ (c) $\dfrac{\sigma R}{r}\hat{\mathbf{r}}$, 0 (d) $\dfrac{q}{2}\hat{\mathbf{r}}$ (e) $q\hat{\mathbf{r}}$

$\mathbf{E}=\dfrac{\mathbf{D}}{\varepsilon_0}$ in all cases

1.4

$$\frac{qr}{4\pi\varepsilon_0 a^3}\hat{\mathbf{r}} \quad 0<r<a$$
$$\frac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \quad r\ge a$$

1.5

(a) $\dfrac{-q}{4\pi b^2}$ (b) $\dfrac{\sigma b^2}{\varepsilon_0 r^2}\hat{\mathbf{r}}$

1.6

(b) $8\pi a^3 K/15$ (b)
$$\frac{K_0 r}{3\varepsilon_0}\left(1-\frac{3r^2}{5a^2}\right)\hat{\mathbf{r}} \quad r<a$$
$$\frac{2K_0 a}{15\varepsilon_0}\hat{\mathbf{r}} \quad r=a$$
$$\frac{2K_0 a^3}{15\varepsilon_0 r^2}\hat{\mathbf{r}} \quad r>a$$

A.2

2.6

(a) $-Kr^2\hat{\mathbf{r}}+3Krz\hat{\mathbf{z}}$, $\mathbf{E}$ is not conservative (b) $-12K\pi$

2.7

(a) 21 (b) 0

2.8

-10, -6 (**E** is not conservative)

2.9

(a) $12\pi(k_1+2k_2)$

2.10

(a)

$$-V_0(y\hat{\mathbf{x}}+x\hat{\mathbf{y}})$$

$$-\frac{V_0\pi}{a}\left[\sinh\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)\hat{\mathbf{x}}+\cosh\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right)\hat{\mathbf{y}}\right]$$

$$-\frac{V_0\pi}{a}\left[\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right)\hat{\mathbf{x}}-\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)\hat{\mathbf{y}}\right]$$

(b) 0, 0, $\dfrac{2\varepsilon_0 V_0\pi^2}{a^2}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right)$

2.11

(a) 0, 3, 2 (b) $3xy-2yz-\dfrac{z^2}{2}$

2.12

(a) 0 (c) Yes

2.13

$a^2\,L/2$