Lecture 4B – Magnetic Circuits

The magnetic circuit. Magnetic and electric equivalent circuits. DC excitation. AC excitation. Characteristics. Determining F given ϕ. Determining ϕ given F (load line). Equivalent circuit of a permanent magnet.
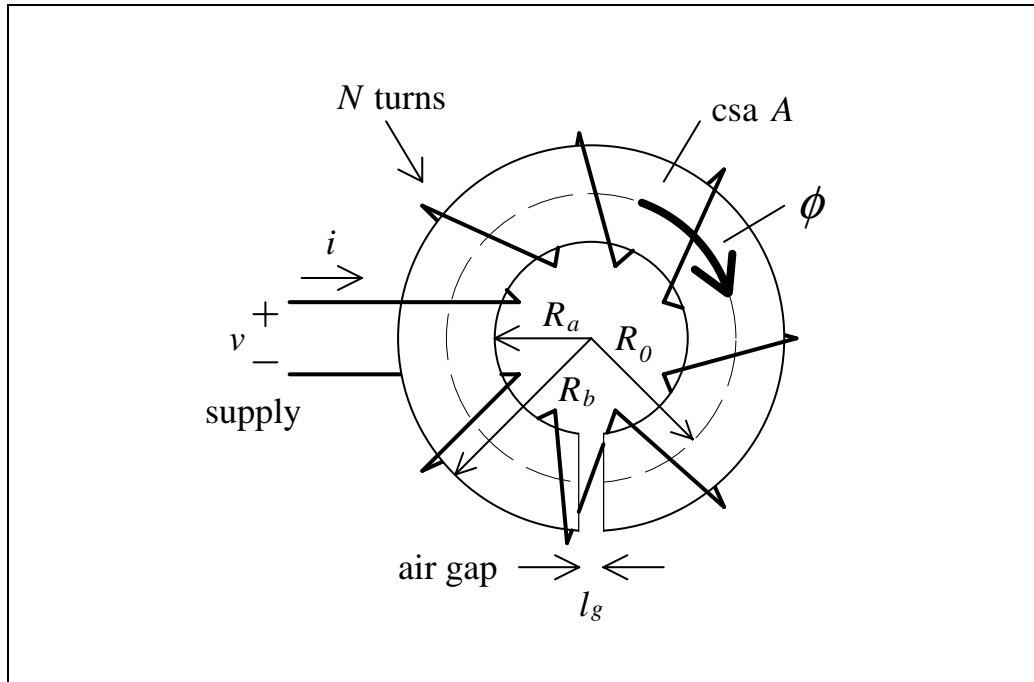
The Magnetic Circuit

Figure 4B.1

Consider a toroid with a core of ferromagnetic material. A small gap is made in the core. We know from previous analysis and by demonstration, that the magnetic field inside a toroid is fairly uniform. No magnetic field lies outside the toroid. For this case, the flux path is defined almost exactly.

The magnetic field in a "thin" toroid is uniform

The ferromagnetic material can be considered a good "conductor" of flux, just like a metal wire is a good conductor of charge. The surrounding air, because of its low permeability, acts like an insulator to the flux, just like ordinary insulation around a metal wire. Since the flux path is well defined, and since the magnetic field is assumed to be uniform, Ampère's Law will reduce to a simple summation.

An analogy between magnetic and electric circuits

4B.2

If the air gap is small, there will not be much fringing of the magnetic field, and the cross section of the air that the flux passes through will be approximately equal to the core cross section.

Let the cross sectional area of the toroid's core be A. The mean length of the core will be defined as the circumference of a circle with a radius the average of the inner and outer radii of the core:

$$l = 2\pi R_0 = 2\pi \frac{R_a + R_b}{2} \tag{4B.1}$$

Ampère's Law around the magnetic circuit gives:

$$\oint_l \mathbf{H} \cdot d\mathbf{l} = Ni$$

$$\oint_l H dl = \sum_l Hl = \sum_l U = Ni \tag{4B.2}$$

The integral can be simplified to a summation, since the field $\mathbf{H}$ is in the same direction as the path $\mathbf{l}$. This is a direct result of having a ferromagnetic material to direct the flux through a well defined path. In all cases we will let the subscript i mean iron (or any ferromagnetic material) and subscript g mean gap (in air). Ampère's Law written explicitly then gives:

$$Ni = H_i l_i + H_g l_g$$

$$= \frac{B_i}{\mu_i} l_i + \frac{B_g}{\mu_0} l_g$$

$$= \frac{l_i}{\mu_i A_i} \phi + \frac{l_g}{\mu_0 A_g} \phi \tag{4B.3}$$

This looks like the magnetic analog of KVL, taken around a circuit consisting of a DC source and two resistors. We will therefore exploit this analogy and develop the concept of *reluctance* and *mmf*.

Define reluctance as:

$$\mathsf{R} = \frac{l}{\mu A}$$

(4B.4)

Reluctance defined

and magnetomotive force (mmf) as:

$$\mathsf{F} = Ni$$

(4B.5) Magnetomotive force (mmf) defined

then Ampère's Law gives:

$$\mathsf{F} = \left(\mathsf{R}_i + \mathsf{R}_g\right)\phi$$
$$= \mathsf{R}\phi$$

(4B.6)

Ampère's Law looks like a "magnetic Ohm's Law" for this simple case

This is analogous to Ohm's law. It should be emphasised that this is only true where μ is a constant. That is, it only applies when the material is linear or assumed to be linear over a particular region.

The inductance of the toroidal coil is given by the definition of inductance:

$$L = \frac{\lambda}{i} = \frac{N\phi}{\sum_l Hl \Big/ N} = \frac{N^2\phi}{\mathsf{R}\phi} = \frac{N^2}{\mathsf{R}}$$

(4B.7)

The inductance of a toroid using physical characteristics

Electromechanical devices have one magnetic circuit and at least one electric circuit. The magnetic material serves as a coupling device for power. Such devices include the transformer, generator, motor and meter.

Because magnetic circuits containing ferromagnetic materials are nonlinear, the relationship $\mathsf{F} = \mathsf{R}\phi$ is not valid, since this was derived for the case where μ is a constant.

Ampère's Law, on the other hand, is always valid, and the concept of magnetic potential will be used where μ is nonlinear.

4B.4

Gauss' Law is the magnetic analog of KCL

The magnetic analog to KVL is Ampère's Law. *What is the magnetic analog to KCL?* In simple systems where the flux path is known, the flux entering a point must also leave it. The analog to KCL for magnetic circuits is therefore Gauss' Law.

The two laws we will use for *magnetic circuits* are:

Ampère's Law and Gauss' Law are simple summations for magnetic circuits

$$\sum_l \mathsf{F} = \sum_l U, \quad \text{around a loop } l \qquad (4B.8a)$$

$$\sum \phi = 0, \quad \text{at a node} \qquad (4B.8b)$$

Magnetic and Electric Equivalent Circuits

All electromagnetic systems should be reduced to simple magnetic and electric equivalent circuits

To formalise our problem solving capabilities, we will convert every conceivable electromagnetic device into an equivalent magnetic circuit and an equivalent electric circuit. We can analyse such circuits using techniques with which we are familiar. The magnetic circuit for the toroidal coil is:
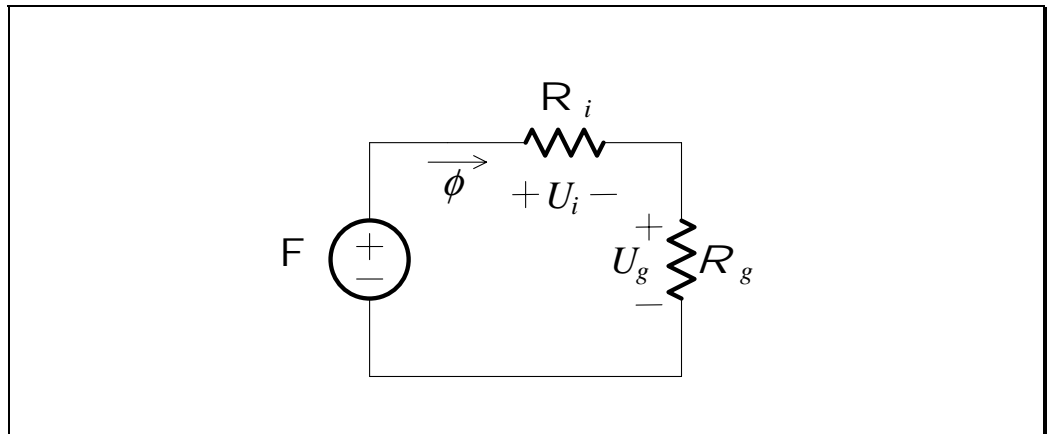

Figure 4B.2

There are various ways to analyse the circuit, depending on whether we know the current or flux, but all methods involve Ampère's Law around the loop:

$$\begin{aligned} \mathsf{F} &= U_i + U_g \\ &= H_i l_i + H_g l_g \end{aligned} \qquad (4B.9)$$

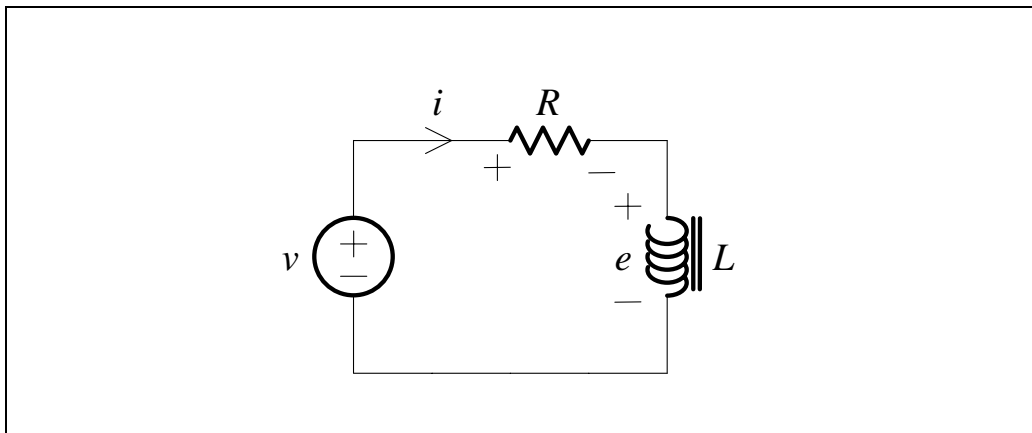The electric circuit for the toroidal coil is:

Figure 4B.3

KVL around the loop gives:

$$v = Ri + e \qquad \text{(4B.10)}$$

It is normally a difficult circuit to analyse because of the nonlinear inductance (which must be taken from a λ-i characteristic).

The voltage source applied to the coil is said to *excite* the coil, and is known as voltage *excitation*. Two special cases of excitation are of particular interest and practical significance – DC excitation and AC excitation.

DC Excitation

DC excitation refers to the case where a source is applied to the coil which is constant with respect to time.

DC excitation defined

For DC excitation, in the steady-state, the electric circuit is easy to analyse. Faraday's Law for the inductor is:

$$e = \frac{d\lambda}{dt} = \frac{d(Li)}{dt} = L\frac{di}{dt} \qquad \text{(4B.11)}$$

where L may be nonlinear. Initially, the circuit will undergo a period of *transient* behaviour, where the current will build up and gradually converge to a *steady-state* value. The circuit will be in the steady-state when there is no

4B.6

more change in the current, i.e. $di/dt = 0$. Then Faraday's Law gives the voltage across the inductor as 0 volts (regardless of whether the inductance is linear or not).

KVL around the equivalent circuit then gives:

$$V = RI \qquad \text{(4B.12)}$$

where we use a capital letter for *I* to indicate a constant, or DC, current.

Therefore, there is a direct relationship, in the form of Ohm's Law, between the applied voltage and the resultant steady-state current, i.e. the voltage source sets the *current*, so we need to look up the resultant *flux* on the inductor's $\lambda \sim i$ characteristic. This flux is obviously constant with respect to time, since the current is constant with respect to time.

AC Excitation

AC excitation refers to the case where a source is applied to the coil which is continuously changing with respect to time – in most cases the excitation is sinusoidal.

For AC excitation, we simplify the analysis by assuming the resistance is negligible. KVL then gives:

$$v = \hat{V} \cos(\omega t) = e = \frac{d\lambda}{dt} \qquad \text{(4B.13)}$$

and:

$$\lambda = \int_0^t e\, d\tau = \int_0^t \hat{V} \cos(\omega\tau)\, d\tau = \frac{\hat{V}}{\omega} \sin(\omega\tau) \qquad \text{(4B.14)}$$

Therefore, there is a direct relationship between the applied voltage and the resultant sinusoidal flux, i.e. the voltage source sets the *flux*, so we need to look up the resultant *current* on the inductor's $\lambda \sim i$ characteristic. Even though the

flux is sinusoidal, the resulting current is not sinusoidal, due to the hysteresis characteristic of the ferromagnetic material used to make the inductor. However, the current is periodic, and it does possess *half-wave symmetry*.

Characteristics

The *B-H* characteristic can be converted to a ϕ-*U* characteristic for a given material:

A *B-H* characteristic can be rescaled to give a ϕ-*U* characteristic

$$\phi = BA$$
$$U = Hl$$

(4B.15)

For this particular case, there is only one path that the flux takes, so the flux is the same through each material (iron and air). We should, for a given flux, be able to look up on each material's characteristic how much *U* there is because of this flux. The total *U* for the magnetic circuit for a given flux is just the addition of the two *U*s. For each value of flux, we can draw the corresponding total value of *U*. The result is a *composite characteristic*. It is useful if there is more than one ferromagnetic material in the circuit.

How a "composite" *B-H* characteristic takes into account the properties of all the materials in a magnetic circuit
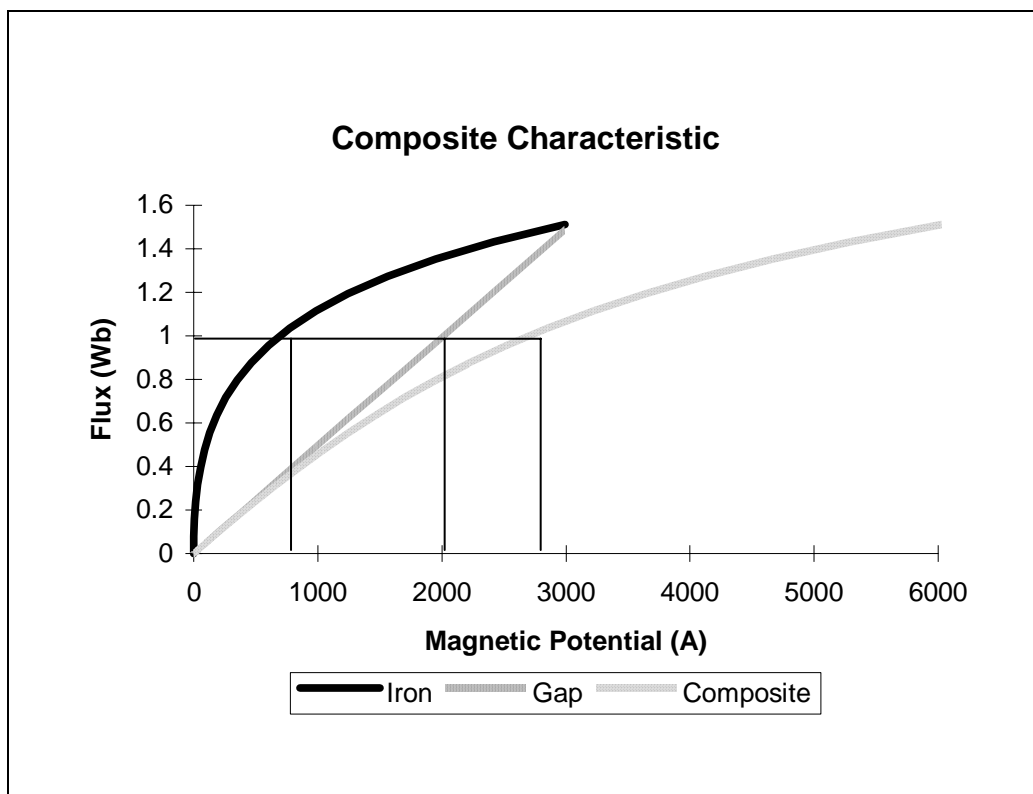

Figure 4B.4

4B.8

Determining F given ϕ

If we are given the flux, then the potentials can be obtained from a ϕ-U characteristic.
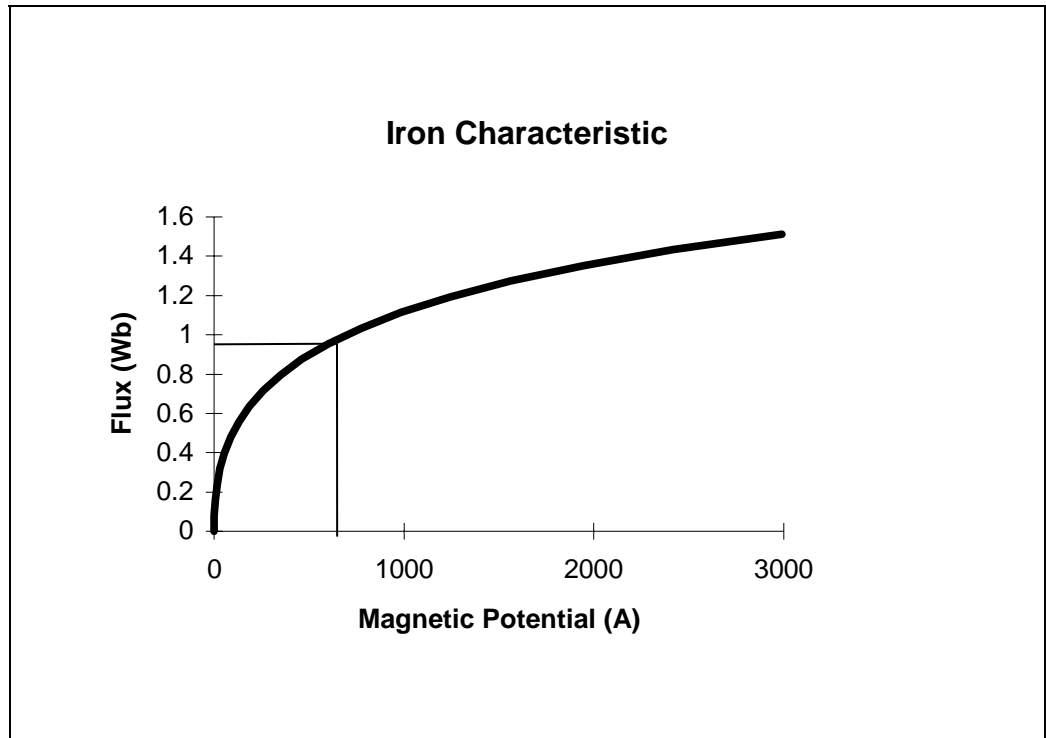
Iron Characteristic

Figure 4B.5

For air gaps, we don't need a characteristic, since it is linear. We use:

For air gaps, the characteristic is a straight line, so use the equation

$$U_g = \mathsf{R}_g\phi = \frac{l_g}{\mu_0 A_g}\phi \qquad (4\text{B}.16)$$

Ampère's Law is applied, and we get:

$$\mathsf{F} = U_i + U_g = Ni \qquad (4\text{B}.17)$$

Given F, the choice of N and i is dictated by other considerations

The number of turns and current can be chosen to suit the physical conditions, e.g. small wire (low current rating) with lots of turns or large wire (high current rating) with a few turns.

Determining ϕ given F (Load Line)

An iterative procedure may be carried out in this case. A better way is to use a concept called the load line. (A gap is said to "load" a magnetic circuit, since it has a high U). This concept is used in graphical analysis of nonlinear systems. The load line, being linear, must be derived from a linear part of the system. In a magnetic circuit, the air gap has a linear relationship between ϕ and U.

Using Ampère's Law, we get:

$$\mathsf{F} = U_i + U_g = U_i + \mathsf{R}_g \phi$$

$$\phi = -\frac{1}{\mathsf{R}_g}(U_i - \mathsf{F})$$

The load line equation for a magnetic circuit with one gap

(4B.18)

This is the equation of the load line. The unknown quantities are ϕ and U_i. This equation must be satisfied at all times (Ampère's Law is always obeyed). There are two unknowns and one equation. *How do we solve it?*

We need another equation. The other equation that must be obeyed at all times is one which is given in the form of a graph – the material's characteristic. It is nonlinear.

4B.10

To solve the system of two equations in two unknowns, we plot the load line on the characteristic. Both graphs are satisfied at the point of intersection. We can read off the flux and potential.
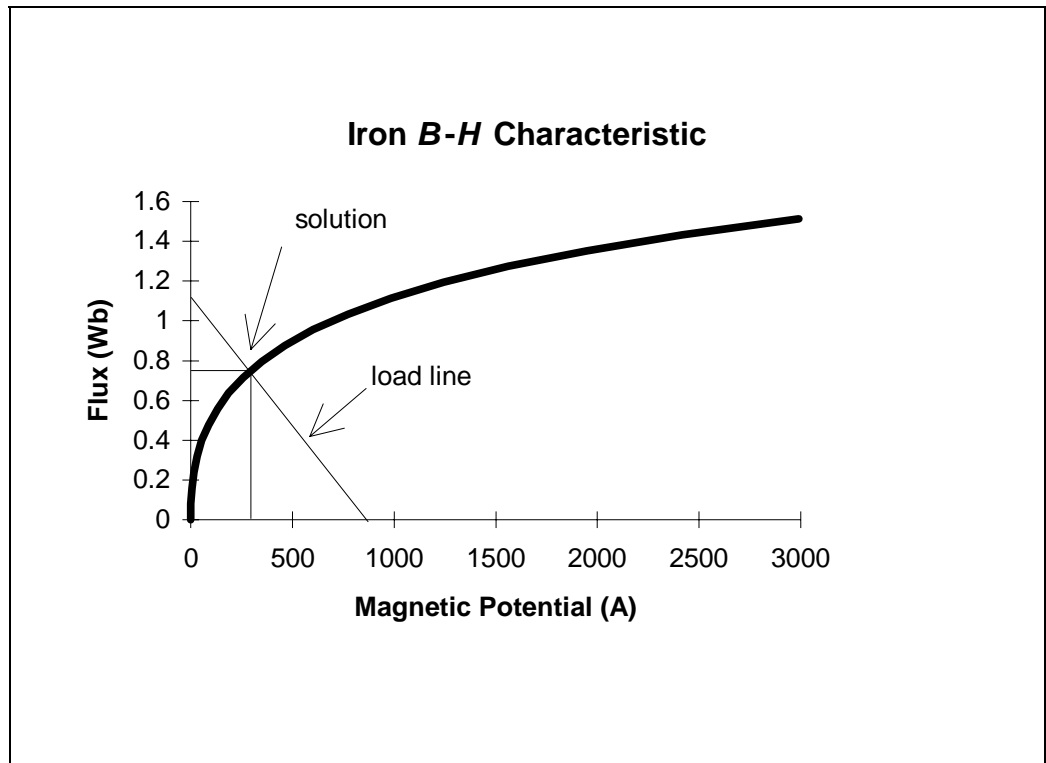

Figure 4B.6

If the material is specified in terms of a *B-H* characteristic, then the equation of the line becomes:

$$\mathsf{F} = H_i l_i + \frac{B_g}{\mu_0} l_g$$

$$B_g = -\frac{\mu_0 l_i}{l_g}\left(H_i - \frac{\mathsf{F}}{l_i} \right) \qquad (4B.19)$$

Equivalent Circuit of a Permanent Magnet
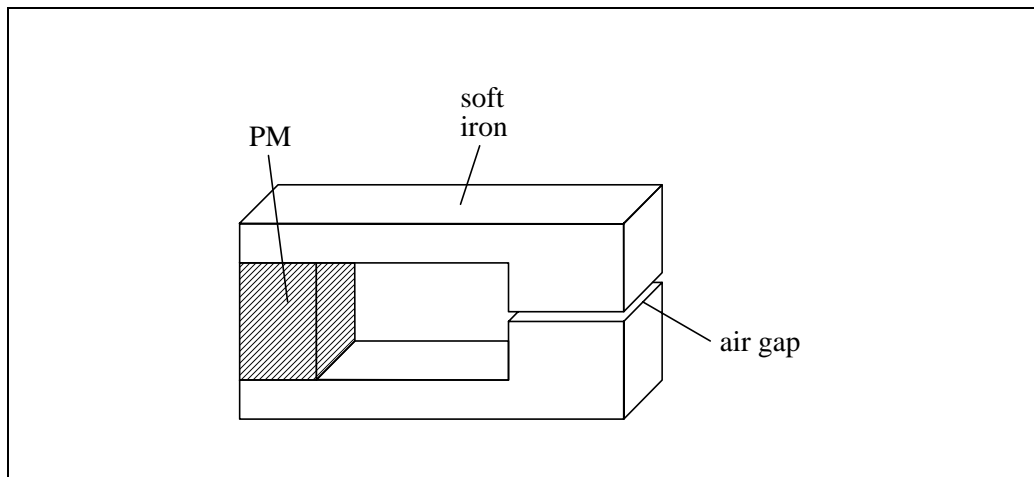
Figure 4B.7

In this case, we know the mmf – it is zero since there is no applied current. The method of finding the flux in a magnetic circuit containing a permanent magnet (PM) therefore follows the same procedure as above. We ignore the soft iron (it has infinite permeability compared to the air gap):

A permanent magnet (PM) produces flux without mmf

$$0 = U_m + U_g = U_m + \mathsf{R}_g \phi$$

$$\phi = -\frac{1}{\mathsf{R}_g} U_m$$

(4B.20)

Ampère's Law (load line equation) for a PM

To solve for the flux, we need the PM's characteristic. We can see that for a positive flux, the magnet exhibits a negative potential. This makes sense because we have always assumed that the magnetic potential is a drop. A negative drop is equivalent to a rise – a PM is a source of potential and therefore flux. The load line intersects the *B-H* hysteresis loop (not the normal magnetization characteristic) to give the operating point (or *quiescent* point, or *Q*-point for short).
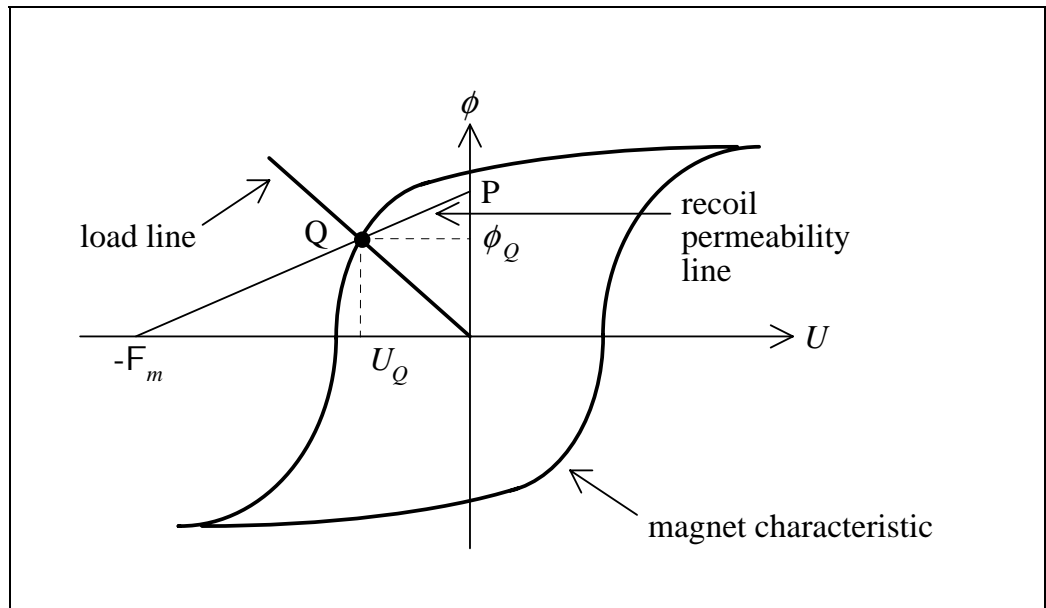
For a PM, only one part of the hysteresis loop is valid

4B.12

Figure 4B.8

A PM exhibits hysteresis, so when the gap is replaced with a soft iron keeper, the characteristic is not traced back. The operating point moves along another line called the recoil permeability line (PQ in Figure 4B.8) to P. Subsequent opening and closing of the gap will cause the operating point to move along PQ. A good permanent magnet will operate along PQ almost continuously.

If the operating point always lies between P and Q, then we can use the equation for this straight line in the analysis. This is also equivalent to modelling the PM with linear elements.
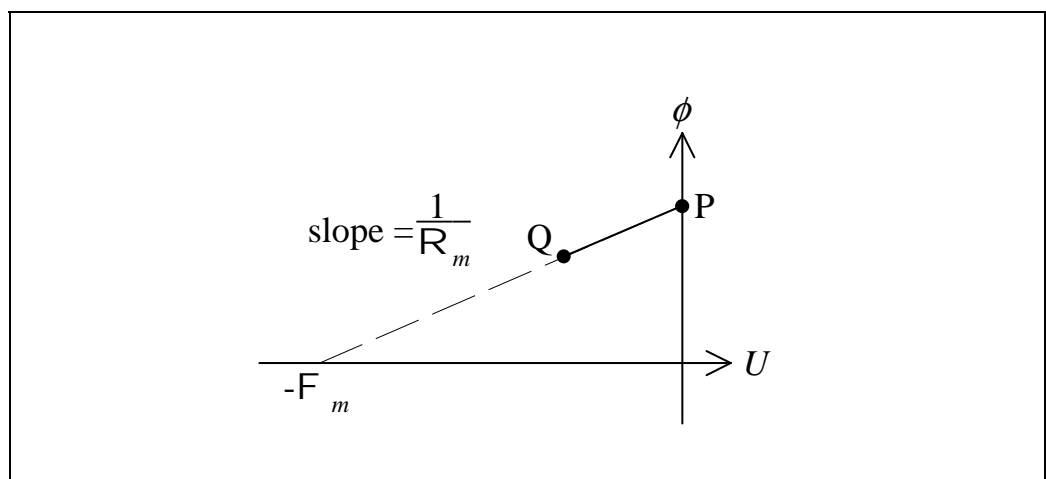
A linear model of a PM hysteresis loop

Figure 4B.9
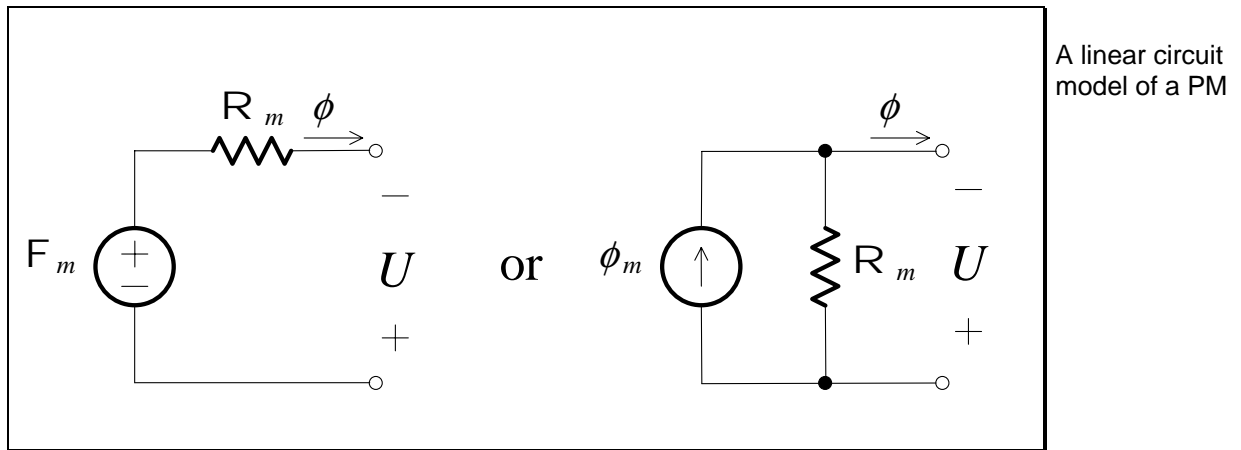
The PM linear model is therefore:

A linear circuit model of a PM

Figure 4B.10

The linear circuit model for the PM is only valid for load lines that cross the recoil line between P and Q.

4B.14

Example – Determine F given ϕ
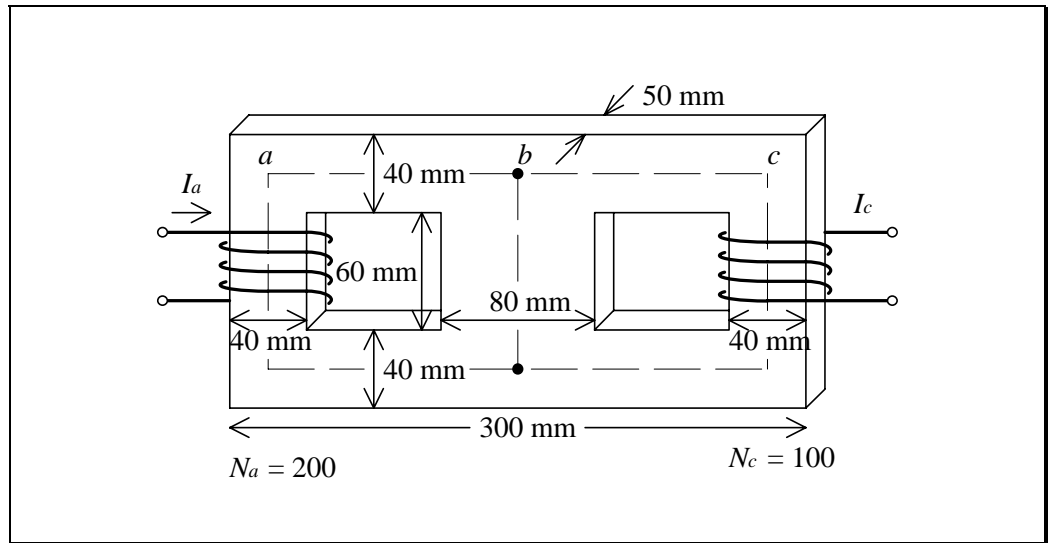
Consider the following electromagnetic system:

Figure 4B.11

Given:

- The core is laminated sheet steel with a stacking factor = 0.9.

- $\phi_a = 1.8$ mWb, $\phi_b = 0.8$ mWb, $\phi_c = 1$ mWb.
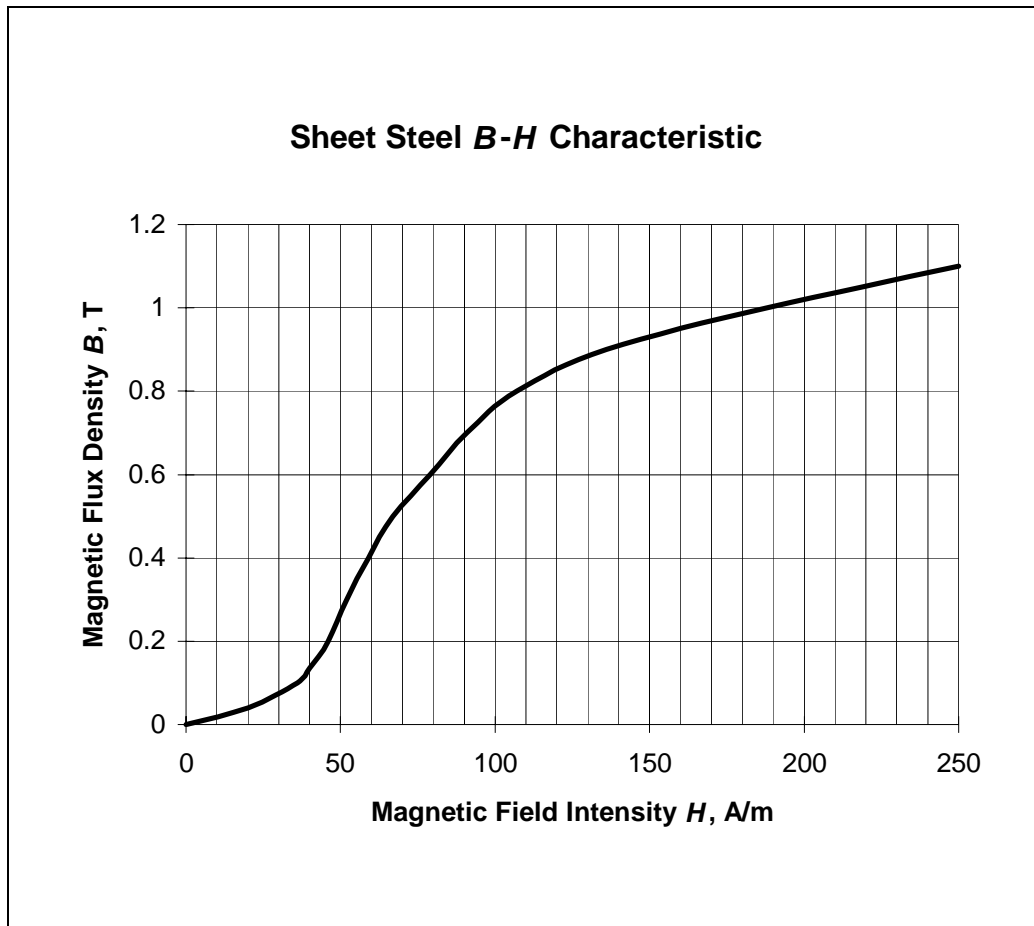
- I_a is in the direction shown.

Sheet Steel *B-H* Characteristic

Figure 4B.12

Draw the magnetic equivalent circuit.

Show the directions of ϕ_a, ϕ_b and ϕ_c.

Determine the magnitude of I_a and the magnitude and direction of I_c.

4B.16

Solution:

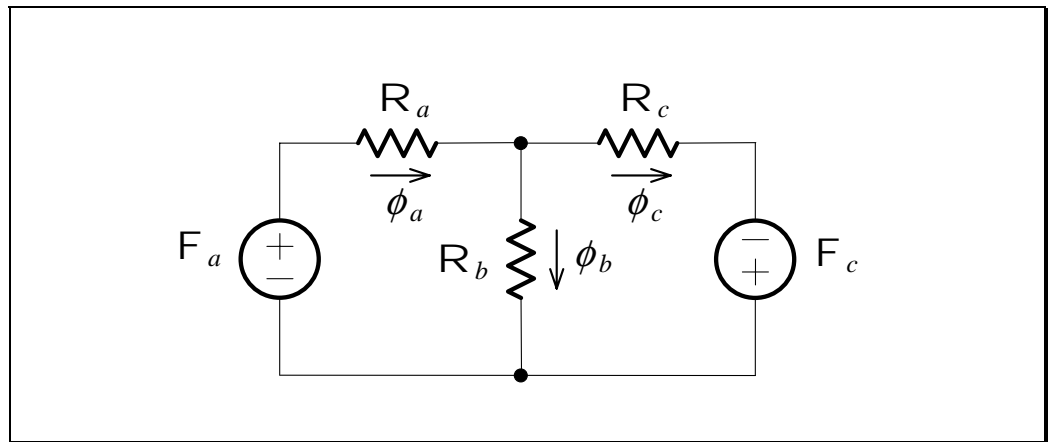The magnetic equivalent circuit is:

Figure 4B.13

As the cross sectional area is uniform, branches a and c are taken right up to the middle of the centre limb. Therefore: $l_a = 0.36$ m, $l_b = 0.1$ m and $l_c = 0.36$ m.

From the *B-H* characteristic, since $B = \phi / A$ and ϕ is given, we just look up:

$$H_a = 200 \text{ Am}^{-1}, \ H_b = 50 \text{ Am}^{-1} \text{ and } H_c = 75 \text{ Am}^{-1}.$$

Applying Ampère's Law around the left hand side (LHS) loop gives:

$$\mathsf{F}_a = U_a + U_b = H_a l_a + H_b l_b = 200 \times 0.36 + 50 \times 0.1 = 77 \text{ A}$$

Applying Ampère's Law around the right hand side (RHS) loop gives:

$$\mathsf{F}_c = U_c - U_b = 75 \times 0.36 - 50 \times 0.1 = 22 \text{ A}$$

Therefore: $I_a = \mathsf{F}_a / N_a = 0.385$ A and $I_c = \mathsf{F}_c / N_c = 0.22 \text{ A} \, (\rightarrow)$.

Example – Permanent Magnet Operating Point
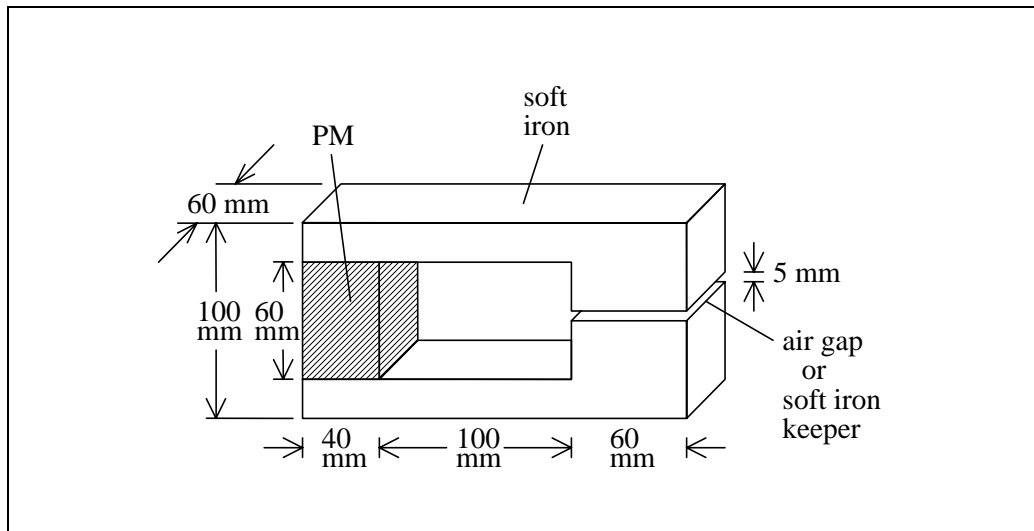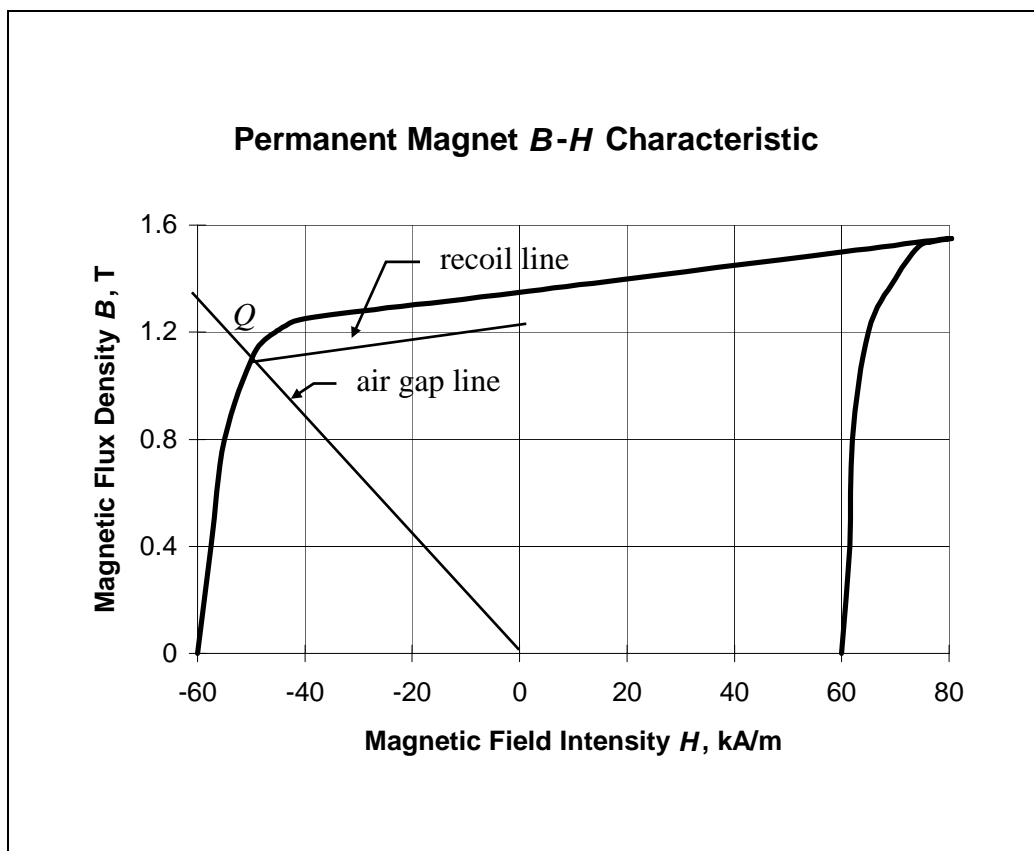
Consider the following permanent magnet (PM) arrangement:

Figure 4B.14

The PM has the following *B-H* characteristic:

PMs "operate" with
a negative **H**

Figure 4B.15

4B.18

a) Remove keeper. Determine gap flux density B_g.

b) Insert keeper. Determine residual flux density.

Ignore leakage and fringing flux.

Assume $\mu_{recoil} = 2\mu_0$ and $\mu_{soft\ iron} = \infty$.

$m \equiv$ magnet, $i \equiv$ soft iron, $g \equiv$ gap.

Solution:

a) As there is no externally applied current, Ampère's Law gives:

$$U_m + \overbrace{U_i}^{= 0 \text{ (since } \mu_i = \infty)} + U_g = 0$$

Therefore:

$$U_m = -U_g = -\mathsf{R}_g \phi \text{ or}$$

The "air gap" line for a PM

$$\phi = -\frac{1}{\mathsf{R}_g} U_m \qquad (4\text{B}.21)$$

i.e. the equation of a straight line (the *air gap line* or *load line*).

If the *B-H* characteristic of the PM is re-scaled to give a ϕ_m-U_m characteristic, the load line (slope $= -1/\mathsf{R}_g$) intersects it at the "operating point" Q. Otherwise, as $\phi = BA$ and $U = Hl$:

rewritten in a form suitable for plotting on a *B-H* characteristic

$$B_m = -\mu_0 \frac{A_g l_m}{A_m l_g} H_m \qquad (4\text{B}.22)$$

From the dimensions given (and $\mu_0 = 4\pi \times 10^{-7}$ Hm^{-1}) we get $B_m = -72\pi \times 10^{-7} H_m$. Therefore, at $H_m = -60 \times 10^3$ Am^{-1}, $B_m = 1.355$ T. We draw the load line through this point and the origin. At the operating point Q, $B_m = 1.12$ T. Therefore $B_g = \frac{A_m}{A_g} B_m = \frac{2}{3} B_m = 0.745$ T.

b) When the keeper is reinserted:

We draw a line from the Q-point, with slope $2\mu_0$ to the B axis. The intersection gives the residual flux density B_m.

or:

The change in magnetic field intensity is $\delta H_m = 50 \times 10^3$. The flux density change is $\delta B_m = 2\mu_0 \delta H_m = 0.13$ T. Therefore residual $B_m = 1.12 + 0.13 = 1.25$ T.

Example – Permanent Magnet Minimum Volume

Consider the PM and characteristic of the previous example.

The flux in the PM is:

$$B_g A_g = B_m A_m$$

and the potential across the PM is:

$$H_g l_g = -H_m l_m$$

The volume of the PM can therefore be expressed as:

$$V_m = A_m l_m = l_g A_g \frac{H_g B_g}{\left| H_m B_m \right|}$$

(4B.23)

The volume of a PM can be minimised by careful choice of the Q-point

which is a minimum if $\left| H_m B_m \right|$ is a maximum.

4B.20

If we plot B-$|HB|$ for the magnet we get:

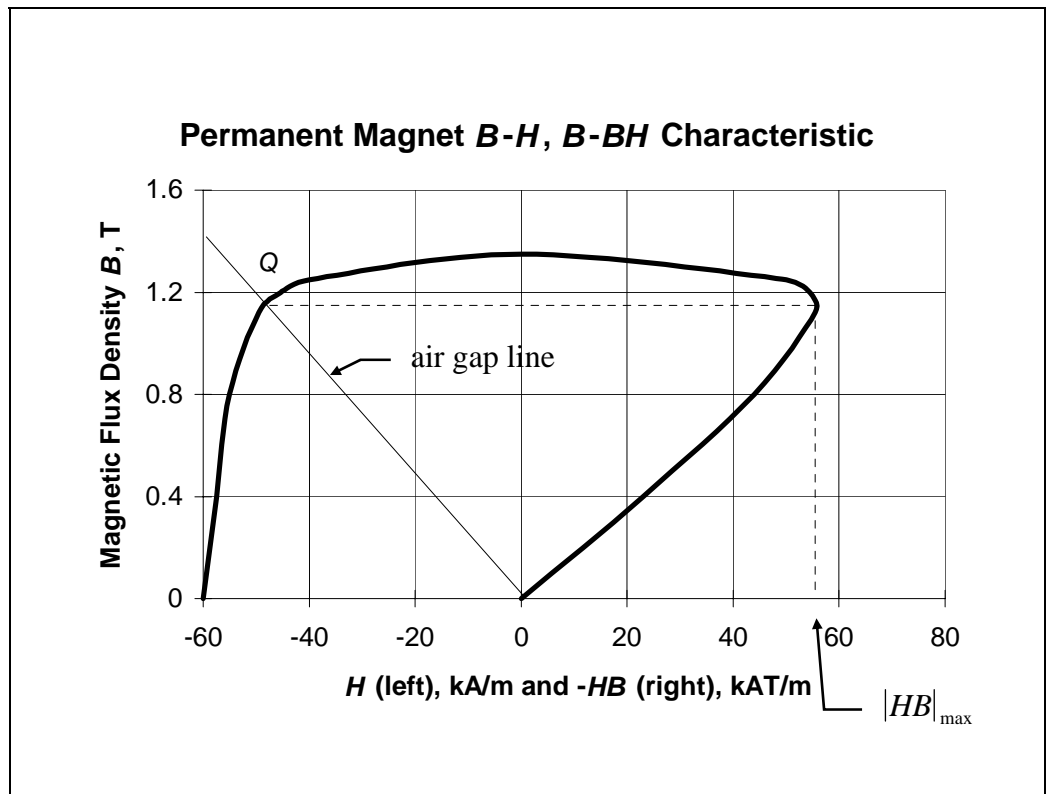**Permanent Magnet B-H, B-BH Characteristic**

Figure 4B.16

Figure 4B.16 shows how the plot is constructed from the PMs B-H characteristic. The plot shows that if we choose a PM with minimum volume (small cost – PMs are expensive) then we should choose an operating point given by Q. The previous example was therefore a good design.

Summary

- We define the reluctance of a magnetic material as: $\mathsf{R} = \dfrac{l}{\mu A}$.

- Wed define the magnetomotive force (mmf) of a coil as: $\mathsf{F} = Ni$.

- For uniform magnetic fields and linear magnetic material, Ampère's Law is the magnetic analog of Ohm's Law: $\mathsf{F} = \mathsf{R}\phi$.

- The inductance of a toroidal coil is given by: $L = \dfrac{N^2}{\mathsf{R}}$.

- Ampère's Law and Gauss' Law are the magnetic analogs of Kirchhoff's Voltage Law and Kirchhoff's Current Law, respectively:
$$\sum_l \mathsf{F} = \sum_l U \text{ and } \sum \phi = 0.$$

- We can convert every electromagnetic device into an equivalent magnetic circuit and an equivalent electric circuit.

- For ferromagnetic materials, we use the nonlinear *B-H* characteristic analyse the magnetic circuit.

- A load line represents a linear relationship between circuit quantities and is usually graphed on a circuit element's characteristic to determine the operating point, or *Q*-point.

- A permanent magnet exhibits an internal negative magnetic potential and can be used to create flux in a magnetic circuit without the need for an external mmf. The operating point of a magnetic circuit that uses a permanent magnet can be optimised to minimise the volume of permanent magnetic material.
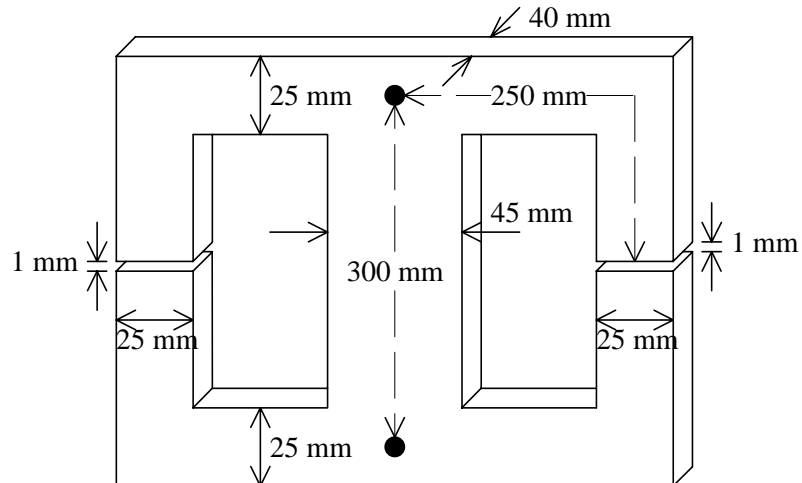
References

Plonus, Martin A.: *Applied Electromagnetics*, McGraw Hill Kogakusha, Ltd., Singapore, 1978.

4B.22

Problems

1.

Consider the following magnetic structure:

The normal magnetisation characteristic of the core material is,

$H\,(\mathrm{Am}^{-1})$	100	200	280	400	600	1000	1500	2500
$B\,(\mathrm{T})$	0.51	0.98	1.20	1.37	1.51	1.65	1.73	1.78

Determine the flux density in the centre limb and the necessary mmf for a winding on the centre limb:

(a)   For a flux density of 1.2 T in each air gap,

(b)   For a flux density of 1.2 T in one air gap when the other is closed with a magnetic material of the same permeability as the core material.

2.

Consider the magnetic structure of Q1. The centre limb has a winding of 500 turns, carrying 1 A.

Draw the equivalent magnetic circuit and determine the total reluctance of the circuit and the flux density in the RHS air gap for:
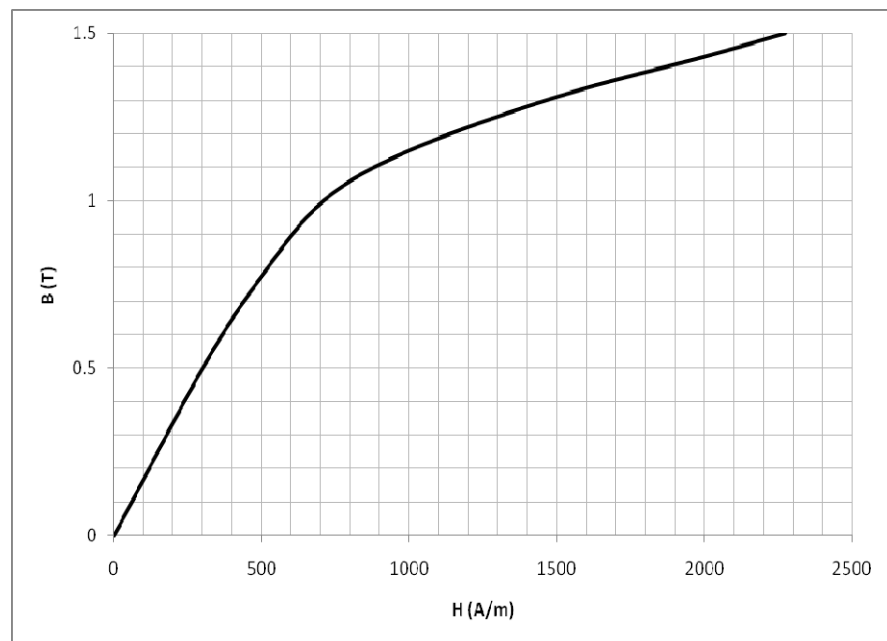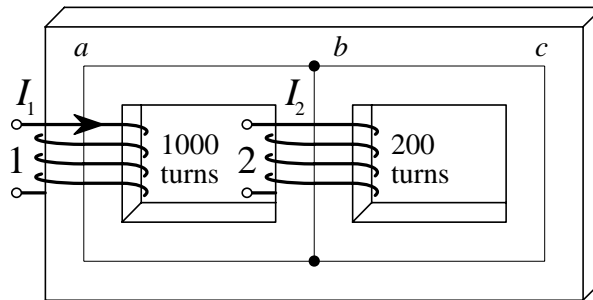
(a)   Both air gaps open,

(b)   LHS air gap closed.

Assume a constant permeability $\mu = 5 \times 10^{-3}$ Hm^{-1}.

4B.24

3.

Consider the magnetic core and magnetization characteristic shown.

The core has a uniform cross sectional area (csa) $A = 6.4 \times 10^{-3}$ m^2, $l_a = l_c = 0.88$ m and $l_b = 0.16$ m. Coil 1 has $I_1 = 0.5$ A (DC).

(a) Determine the magnitude and direction of I_2 needed to give $\phi_b = 0$.

(b) Develop expressions for L_{21} and L_{12}, and calculate their value for the currents and fluxes determined in (a).

4.

Refer to the example *Permanent Magnet Operating Point*.

The air gap flux was assumed to be confined within the air gap. Consider now the leakage flux between the upper and lower horizontal sections of the soft magnetic material, and assume that the leakage flux density is uniform.

(a) Determine the flux density in the PM when the keeper is removed. Compare the air gap flux density with that calculated in the example.

(b) The leakage flux may be reduced by placing the PM closer to the air gap. Sketch an improved arrangement of the system.

4B.26

5.

The PM assembly shown is to be used as a door holder (keeper attached to the door, remainder attached to the frame).
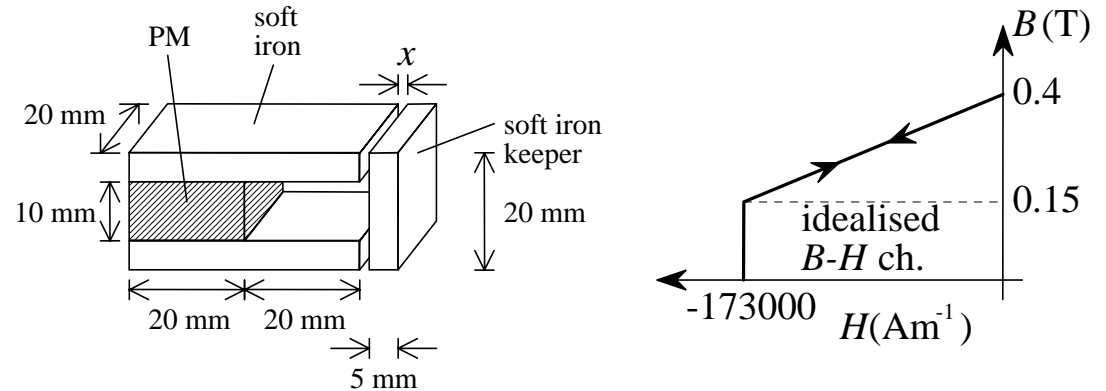
Assume that for soft iron $\mu = \infty$.

(a) Derive a linearised magnetic equivalent circuit (neglect leakage and fringing), and determine the maximum air gap length x_{max} for which it is valid.

(b) The linear model used in (a) assumes that the magnet will be demagnetised when $x > x_{max}$. Show that the leakage reluctance between the upper and lower soft iron pieces is low enough to prevent demagnetisation from occurring.