Lecture 5B – Field Energy

Energy stored in the magnetic field. Electric field energy. Total field energy. Hysteresis losses. Eddy currents.

Energy Stored in the Magnetic Field

The toroid (equivalent to an infinitely long solenoid) is ideal for determining

the magnetising characteristic of specimens (no end effects).



Figure 5B.1

The direction of the field in the iron is given by the right hand screw rule.

KVL gives:

$$v = Ri + e$$

= $Ri + \frac{d\lambda}{dt}$ (5B.1)

Therefore, the electric power delivered by the source is:

$$p = vi = Ri^2 + i\frac{d\lambda}{dt}$$
 W The power delivered to a solenoid

Some of the power delivered to a solenoid is dissipated as heat Positive values for the terms in Eq. (5B.2) represent power delivered by the source. The first term, involving the resistance of the winding, R, is always positive. (*Why?*) This term therefore represents a power dissipation, or loss, in the form of heat that always exists – regardless of the current direction. In words, Eq. (5B.2) reads as:

Since there is no electrical or mechanical output, this equation is really just a statement of the conservation of energy: *energy in = energy out*.

A negative number in Eq. (5B.2) represents power delivered *to* the source. The power associated with the rate of change of flux is therefore not a loss, since it can have a negative value. It represents power either stored or delivered by the field.

Consider first a DC supply, with the current positive. When it is switched on, the magnetic field must increase from 0 to some value. This is a positive rate of change of flux linkage, so the last term in Eq. (5B.2) is positive – power has been delivered by the source to establish the field.

In the steady state, the current will be a constant, so the flux linkage will not change – no more power is delivered to the field and the only loss is resistance.

When switching the supply off, the field must return to zero from a positive value. This is a negative rate of change of flux linkage, so the last term is negative – power has been delivered to the source from the field. The stored field energy is returned to the system in some form. (Note what happens when you switch off an inductive load – the energy stored in the field is dissipated as an arc in the switch).

Some of the energy delivered to a solenoid is stored in the magnetic field

For steady state DC, the energy stored in the field is constant the power to the field is zero

Stored magnetic field energy is released when the current (which causes the field) is interrupted Consider a sinusoidal supply. The power delivered by the source will vary with time. Firstly, consider the instantaneous power in the resistor:

$$p_{R} = Ri^{2}$$
$$= R\hat{I}^{2}\cos^{2}\omega t$$
$$= R\hat{I}^{2}\frac{1}{2}(1+\cos 2\omega t)$$

The instantaneous power delivered to a resistor from a sinusoidal supply

(5B.4)

The average power dissipated in the resistor is:

$$P_{R} = \frac{1}{T} \int_{0}^{T} \frac{R\hat{I}^{2}}{2} (1 + \cos 2\omega t) dt$$
$$= \frac{R\hat{I}^{2}}{2T} \left[t + \frac{1}{2\omega} \sin 2\omega t \right]_{0}^{T}$$
$$= \frac{R\hat{I}^{2}}{2} = RI_{RMS}^{2}$$

(5B.5) The average power delivered to a resistor from a sinusoidal supply

This term is always positive, and should be familiar. Assuming a linear inductance, the instantaneous power delivered *to* the field is:

$$p_{L} = i \frac{d\lambda}{dt}$$
$$= \hat{I} \cos \omega t \frac{d}{dt} (L\hat{I} \cos \omega t)$$
$$= -\omega L\hat{I}^{2} \cos \omega t \sin \omega t$$
$$= -\omega L\hat{I}^{2} \frac{1}{2} \sin 2\omega t$$

The instantaneous power delivered to an inductor from a sinusoidal supply

(5B.6)

We can now derive the average power delivered to the inductor:

$$P_{L} = \frac{1}{T} \int_{0}^{T} \frac{-\omega L \hat{I}^{2}}{2} \sin 2\omega t dt$$
$$= \frac{-\omega L \hat{I}^{2}}{2T} \left[\frac{-1}{2\omega} \cos 2\omega t \right]_{0}^{T}$$
$$= 0$$
(5B.7)

The average power delivered to an inductor from a sinusoidal supply

> The average power is zero. The instantaneous power tells us that power constantly goes to and from the field in a sinusoidal fashion. No power is lost in the field because the *average* power delivered is zero.

> Therefore, we can say that the magnetic field, at any instant, is storing energy away at a rate given by:

$$p_{field} = i \frac{d\lambda}{dt}$$
(5B.8)

The energy stored in the magnetic field, in a given time interval, is:

$$W_f = \int_{t_1}^{t_2} p_{field} dt = \int_{t_1}^{t_2} i \frac{d\lambda}{dt} dt = \int_{\lambda_1}^{\lambda_2} i d\lambda$$
(5B.9)

Applying Gauss' law and Ampère's law around the magnetic circuit, we get:

$$i = \frac{Hl}{N}$$

$$d\lambda = Nd\phi = NAdB$$
(5B.10)

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Instantaneous field power defined

Field energy defined

which means the field energy can be expressed by:

$$W_{f} = \int_{\lambda_{1}}^{\lambda_{2}} id\lambda$$
$$= \int_{B_{1}}^{B_{2}} \frac{Hl}{N} NAdB = lA \int_{B_{1}}^{B_{2}} HdB$$
(5B.11)

The energy per unit volume is therefore:

$$\frac{W_f}{V} = \int_{B_1}^{B_2} H dB \quad \text{Jm}^{-3}$$

(5B.12) Field energy density

Integrating the energy by parts (since current is a function of flux linkage – the magnetisation characteristic), we obtain:

$$W_{f} = \int_{\lambda_{1}}^{\lambda_{2}} i d\lambda$$
$$= [i\lambda]_{\lambda_{1}}^{\lambda_{2}} - \int_{i_{1}}^{i_{2}} \lambda di$$
(5B.13)

The last term in Eq. (5B.13) has the units of energy, but is not directly related to the energy stored by the magnetic field. We define a new quantity called the magnetic field *co-energy* to be:

$$W_f' = \int_{i_1}^{i_2} \lambda di$$

(5B.14) Field co-energy defined

With this definition, and using Eq. (5B.13), we get the relationship:

energy + co-energy =
$$i_2 \lambda_2 - i_1 \lambda_1$$

 $W_f + W'_f = i_2 \lambda_2 - i_1 \lambda_1$ (5B.15)

A graphical interpretation of this relationship is shown below:

Energy and coenergy shown graphically



Figure 5B.2

To determine the total energy stored in a magnetic field, that was brought from zero to a steady value, we set the flux and current to zero at time t_1 :

$$W_{f} = \int_{0}^{\lambda} i d\lambda$$
$$W_{f}' = \int_{0}^{I} \lambda di$$
$$W_{f} + W_{f}' = I\lambda$$
(5B.16)

For linear systems, which have straight line $\lambda - i$ characteristics, the energy and co-energy are equal (they are triangles) and are given by:

$$W_{f} = W_{f}' = \frac{1}{2}I\lambda = \frac{1}{2}LI^{2}$$

(since $\lambda = LI$) (5B.17)

If the soft iron keeper is removed from the toroid to create an air gap, then two different uniform fields are created – one in the iron and one in the gap. Each will have its own energy. Assume that the permeability of the iron is a constant (this corresponds to linearising the B-H characteristic).

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The total energy and co-energy stored in a magnetic system

The energy and coenergy are equal for linear systems – and are easily calculated The energy density of a field with constant permeability is:

$$\frac{W_f}{V} = \int_0^B H dB = \int_0^B \frac{B}{\mu} dB = \frac{B^2}{2\mu} Jm^{-3}$$
(5B.18) The energy density of a linear system

The energies in the gap and iron are therefore:

$$\frac{W_{firon}}{V_{iron}} = \frac{B^2}{2\mu_0\mu_r}$$

$$\frac{W_{fgap}}{V_{gap}} = \frac{B^2}{2\mu_0} = \mu_r \frac{W_{firon}}{V_{iron}}$$
(5B.19) The energy density of an air gap is much larger than the energy density of iron

For practical values of permeability and volume, the energy of the field in the air gap is much larger than that in the iron. This is usually what we want, since it is in the gap where the magnetic field is used (e.g. motor, generator, meter).

Electric Field Energy

The same analysis as above can be performed with a capacitor to calculate the energy per unit volume stored in the electric field:

$$\frac{W_f}{V} = \int_0^D E dD \ \text{Jm}^{-3}$$
 (5B.20) The energy density for electric fields

Total Field Energy

In general, the total energy per unit volume stored in a system is:

$$\frac{W_f}{V} = \int_0^D E dD + \int_0^B H dB \text{ Jm}^{-3}$$
 (5B.21) The energy density considering all fields

Hysteresis Losses

With ferromagnetic cores, when the applied field is varied, energy is dissipated as heat during the realignment of the domain walls and a power loss results.

We know that the field energy density is the area under the *B*-*H* loop. Let's see what happens to the field energy when the operating point goes around the *B*-*H* loop.

Imagine the hysteresis loop has already been established, and we are at point *a*.



Figure 5B.3

At this point, the instantaneous energy supplied to the field is zero. If we increase the applied field strength H to the point b, then the energy supplied to the field is:

$$W_{ab} = V \int_0^{\hat{B}} H dB \tag{5B.22}$$

The area of the *B-H* hysteresis loop represents energy density *loss* in a magnetic system

When the applied field strength is brought back to zero at point c, energy is returned to the source, but there has been some energy loss.

$$W_{bc} = V \int_{\hat{B}}^{B_c} H dB \tag{5B.23}$$

The loss is the diagonally striped part of the hysteresis loop. In one complete cycle, the energy loss is equal to:

$$W_h = V \times (\text{area of } B - H \text{ loop}) \text{ J/cycle}$$
 (5B.24) The energy lost in completing one cycle of the *B-H* hysteresis loop

This is a direct result of the hysteresis *B*-*H* relationship of the specimen.

Eddy Currents

Consider a rectangular cross section core. If the magnetic field is periodically varying in time, then Faraday's Law tells us that a voltage will be induced in the ferromagnetic core, causing a current.

This results in a heat loss. The direction of the induced current will be such as to oppose the change in the flux (Lenz's Law).



Figure 5B.4

Explain why there is a non-uniform distribution of flux, and therefore an inefficiency in core usage. (Saturated at outer edges, not saturated in the centre). Label the direction of the eddy currents.

To overcome this, the core is laminated to break up the eddy current paths. This results in lower losses (due to higher resistance) and better utilisation of the core area for the flux.



Figure 5B.5

The laminations are insulated either with an oxide layer or an enamel or varnish. This means the cross sectional area does not give us the cross sectional area of the ferromagnetic material – it includes the core insulation.

We therefore define a stacking factor, which multiplies the cross sectional area to give the ferromagnetic cross sectional area. e.g. a stacking factor of 0.9 means 90% of the area is ferromagnetic, 10% is insulation.

If the rate of change of flux is large, a large voltage is induced, so eddy currents increase. If the frequency of an alternating flux is large, it tends to increase the area of the B-H loop.

Eddy currents can be reduced by laminating the iron

The cross sectional area of the laminations is not the same as that of the iron – take this into account with a "stacking factor"

Summary

- The power delivered to an inductor is either dissipated as heat or used to store energy in the magnetic field. The stored energy in the magnetic field can be returned to the system.
- Co-energy is a quantity of energy that does not exist anywhere inside (or outside) a magnetic system; nevertheless, it is a useful quantity and will be seen to be related to the magnetic force.
- We can derive field energy density for both the electric and magnetic field.
- With ferromagnetic cores, when the applied field is varied, energy is dissipated as heat during the realignment of the domain walls and a power loss results. The area of the *B-H* hysteresis loop represents energy density loss in a magnetic system.
- If the magnetic field is periodically varying in time, then a voltage will be induced in any nearby ferromagnetic material, causing an eddy current. This results in a heat loss. To minimise this loss, we use laminated ferromagnetic cores.

References

Plonus, Martin A.: *Applied Electromagnetics*, McGraw Hill Kogakusha, Ltd., Singapore, 1978.

Problems

1.

The diagram shows two cast steel cores with the same csa $A = 5 \times 10^{-3} \text{ m}^2$ and mean length l = 500 mm.



- (a) With DC excitation, the flux density in core 1 is $B_1 = 1.4$ T. Determine B_2 , I and the self inductance of the circuit with and without the cast steel stopper in core 2.
- (b) The cast steel stopper is removed and *I* reduced to zero at a constant rate in 25 ms. Calculate the voltage induced in the coil.
- (c) The exciting coil is now connected to a constant voltage, 50 Hz AC supply $(V = \hat{V} \cos \omega t)$. Determine \hat{V} , \hat{I} and \hat{B}_2 if $\hat{B}_1 = 1$ T and the stopper is *in* core 2.
- (d) Determine the field energy and the co-energy in core 2 for case (a).

Determine the field energy and the inductance of the following circuit. The centre limb has a winding of 500 turns, carrying 1 A.



3.

A solenoid (diameter d, length l) is wound with an even number of layers (total number of turns N_1). A small circular coil (N_2 turns, area A) is mounted coaxially at the centre of the solenoid.

Show that the mutual inductance is:

$$M = \frac{\mu_0 N_1 N_2 A}{\sqrt{d^2 + l^2}}$$

Assume the field within the solenoid is uniform and equal to its axial field.

2.

4.

Consider the toroid shown below:



a = 160 mm, b = 120 mm, c = 30 mm

 $\mu_r = 15000, N = 250$

The exciter coil resistance is $r = 15 \Omega$.

- (a) Draw the electric equivalent circuit and determine the inductance *L*.
- (b) An airgap $(l_g = 0.2 \text{ mm})$ is cut across the core. Draw the magnetic and electric equivalent circuits, and determine the inductance representing the airgap. What is the electrical time constant of the circuit?

5.

Consider the toroid of Q4.

If a = 80 mm, b = 60 mm, c = 20 mm, N = 1500 and the normal magnetisation characteristic of the core is:

<i>B</i> (T)	0.01	0.05	0.1	0.2	0.5	0.75	1.0	1.2	1.3	1.4	1.5	1.6
$H\left(\mathrm{Am}^{-1} ight)$	1.1	2.9	4.5	6.4	9.6	12.4	16	22	27.6	37	55	116

Plot:

- (a) $\mu_r \sim B$ and $\mathbb{R} \sim B$ (a few points will be sufficient)
- (b) normal inductance $L \sim i$
- (c) Repeat (a) and (b) with a 0.5 mm airgap cut across the core.

6.

Show that the maximum energy that can be stored in a parallel-plate capacitor is $\varepsilon_r \varepsilon_0 E_b^2/2$ per unit volume (E_b = maximum field strength before breakdown). Compare with the energy stored per unit volume of a lead acid battery.

Typical values:

Material	Permittivity	Breakdown Field Strength				
Air	$\varepsilon_r = 1$	$E_b = 3 \mathrm{kV/mm}$				
Lead Acid	$\varepsilon_r = 3$	$E_b = 150 \mathrm{kV/mm}$				