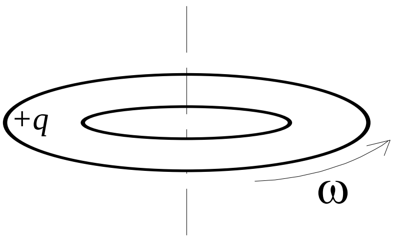
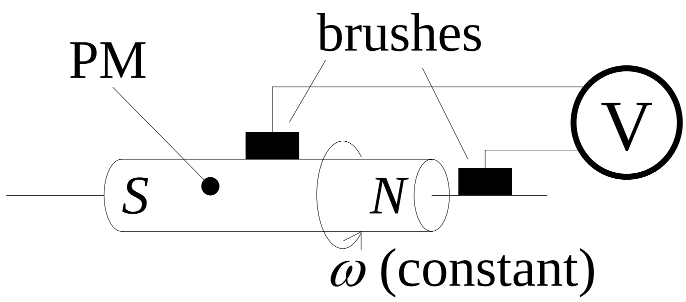
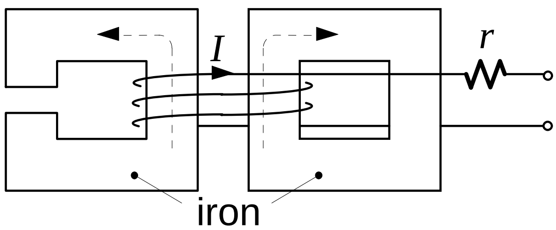
# 48521
# Fundamentals of Electrical Engineering

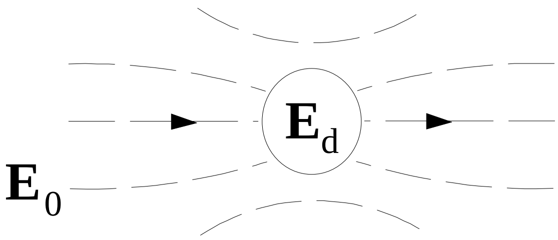
*Lecture Notes*

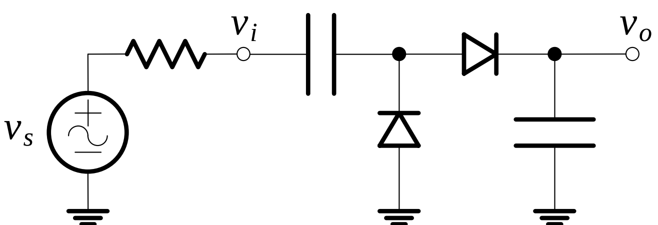
**2010**

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# Contents

Fundamentals of Electrical Engineering 2010

# Lecture 1A – Electrostatics

*Introduction. A brief history of electrostatics. Vectors. The vector dot product. The vector cross product. Area vectors. Coulomb's Law. The electric field. Potential difference. Current density and Ohm's law. Surface integrals. Flux and flux density. Gauss' Law for electrostatics.*

## Introduction

Electrodynamics – the behaviour of moving "charges" – forms the fundamental basis of electrical engineering. We see the effects of electrodynamics daily: lightning and static electricity; magnets and compasses; all the benefits of power systems including domestic, commercial and industrial lighting, heating and the running of motors; telecommunications networks such as radio, television, telephones and the Internet; and the now ubiquitous computer technology. The applications of electrodynamics are diverse (some are simple devices, others are complex systems), but all are described by a few basic principles – it is these *fundamentals laws* that we will study.

Electrodynamics is everywhere

Historically, electric and magnetic phenomena were studied separately. It was only during the 19$^{th}$ century, through the work of several great physicists such as Oersted and Faraday, that a link was found between the two phenomena. Faraday was the consummate experimentalist with a visionary's sense of the unity of nature. He was the first to conceptualize the "electromagnetic field" – a force field that permeates all of space and which gives rise to both electric and magnetic phenomena.

Faraday visualized fields and discovered electromagnetic induction

It then took the genius of James Maxwell to formulate a set of consistent and harmonious mathematical relations between electric and magnetic fields – a unified field theory – which predicted electromagnetic waves and led to the formulation of relativity in the early 20$^{th}$ century by Einstein.

These mathematical relations, now called "Maxwell's equations", successfully describe *all* large-scale electromagnetic phenomena – charged rods, currents in circuits, rotating machines, the way that light propagates through a vacuum, etc. Maxwell's equations are the second most successful equations discovered so far (after the equations of quantum mechanics) in terms of experimental

Maxwell's equations form the basis for all electromagnetic phenomena verification. The equations are used all around us. We live in a world dominated by them – from power generation and the machines that drive industry, to the miniature electronics that has spawned the communication and computing revolution – and so the study of electric and magnetic “fields” is essential for electrical engineers.

Before we embark on a study of electrodynamics, we will firstly consider the much simpler case of *electrostatics*, i.e. the study of electric fields due to static (non-moving) charges.

In retrospect, it is interesting to note that the mathematical equations of *static electric fields*, placed in the framework of the Special Theory of Relativity, also lead to Maxwell’s equations. In this subject, we will follow the historical approach and become familiar with the laws which were postulated based on experimental evidence. The laws in this form are still of great practical use.

## A Brief History of Electrostatics

A brief history of electrostatics

By 600 BCE the ancient Greeks knew that amber (Greek: *elektron*), when rubbed, would attract small quantities of straw, silk, and other light objects. Nothing further was done with this knowledge. Nothing further was learned about electricity for 2200 years.

During the 17th century, there was a lot of attention paid to terrestrial magnetism – because of navigation – and very little to electricity. Scientists were too preoccupied with mechanics and optics (e.g. Newton).

In the 18th century, experiments on frictional electricity became numerous, and the *art* of performing electrical demonstrations developed rapidly. What was still lacking however, was quantitative knowledge of the forces acting between charged bodies. Coulomb provided this in 1785.





## Vectors

The description of electric (and magnetic) phenomena using mathematical equations requires the use of vectors. Vectors portray both the magnitude and direction of a quantity, and are mathematical entities that possess several very important properties. We will revise some of these properties, and introduce new ones as the need arises.

## The Vector Dot Product

The "dot" product of two vectors is defined by:

Dot product defined

$$\mathbf{a}\cdot\mathbf{b} = ab\cos\theta \tag{1A.1}$$

For example, when a force moves an object through a small distance $dl$ it does a small amount of mechanical work $dW$ which is given by:

and used in the definition of work

$$dW = \mathbf{F}\cdot d\mathbf{l} \tag{1A.2}$$

We define the dot product of two vectors this way because the "cos" factor occurs numerous times in the mathematical expressions that describe nature. For example, consider moving a box along the floor:

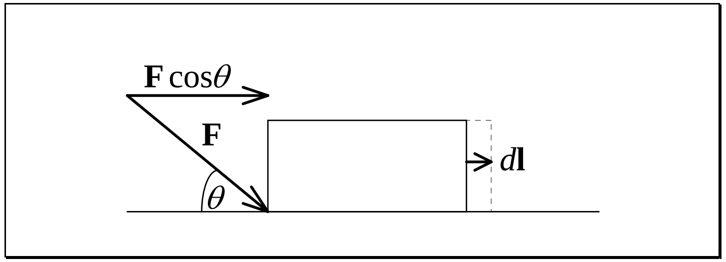


**Figure 1A.1**

Dot product as a shorthand notation

To obtain the work done in moving the box, we first have to figure out the component of the applied force that actually does something useful. That is, we have to find the force that acts in the direction of movement. This "useful" force is shown in the diagram, and you can see that it involves a "cos" term. We can therefore use the shorthand notation of the dot product when writing the expression for the work.

Concept of differentials

Why did we use differentials in Eq. (1A.2)? When something moves, it generally does not follow a straight line. But if we consider very small displacements, (so small that each displacement is a straight line), then Eq. (1A.2) applies. For example, consider pushing an object through an unusual path:

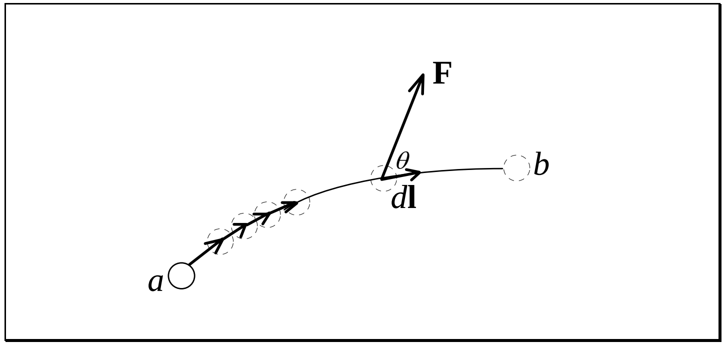


**Figure 1A.2**

Concept of differential work done

An object, when acted upon by a force, is not obliged to move in that direction, as shown in the diagram. We use Eq. (1A.2) to calculate, for each small displacement $d\mathbf{l}$, the small amount of work done in moving the object, $dW$.

To calculate the total work done, in moving the object from point $a$ to point $b$, we perform what is known as a "line" integral (we add up the differentials along a certain curve – in this case the curve $ab$):

Total work done as an integral

$$W = \int_a^b dW = \int_a^b \mathbf{F} \cdot d\mathbf{l} \tag{1A.3}$$





## The Vector Cross Product

The "cross" product of two vectors is defined as:

Cross product defined

$$\mathbf{a} \times \mathbf{b} = ab \sin\theta\, \hat{\mathbf{c}} \tag{1A.4}$$

where the unit vector $\hat{\mathbf{c}}$ has a direction perpendicular to both **a** and **b**. Once again, the definition of the cross product is based upon its utility and frequency in describing natural phenomena.

Right Hand Screw Rule defined

To determine the direction of $\hat{\mathbf{c}}$, we use the *Right Hand Screw Rule*. To apply this rule, you *imagine* the vectors **a** and **b** positioned on a plane. Then "grab hold of" vector **a** and rotate it into vector **b** so that you mimic screwing a lid on a jar, or tightening a right hand screw. The direction of advance of the lid or screw gives the direction of $\hat{\mathbf{c}}$. This should all happen in your mind, *do not* use your hands to perform this mental operation.

The cross product defines a Cartesian coordinate system:

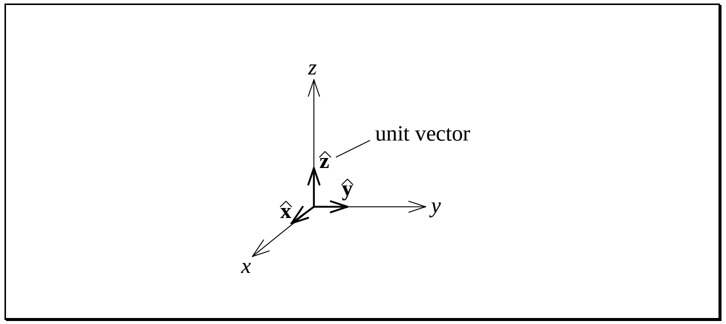


**Figure 1A.3**

Cartesian coordinate system

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}, \quad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \quad \hat{\mathbf{y}} \times \hat{\mathbf{x}} = -\hat{\mathbf{z}} \tag{1A.5}$$

## Area Vectors

Area vector defined

An *area* vector has the job of specifying the size and direction of an area. Direction of an area? Yes, by convention, the *direction* of an area is defined to be the direction perpendicular to the plane of the area. Of course, this implies that the area is flat, but it also applies to curved surfaces that are infinitesimally small.

For example, the area vector for a rectangle would look like:

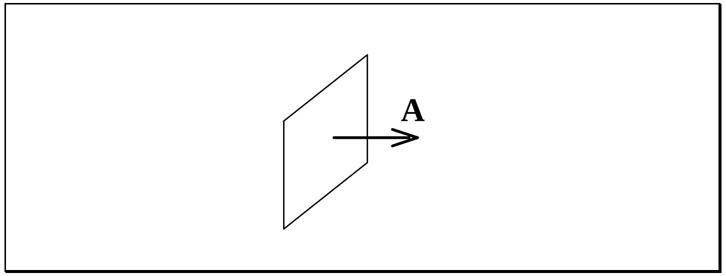


**Figure 1A.4**

Differential area vector

where the *magnitude* of **A** would equal the area of the rectangle. An infinitesimally small area, such as part of a sphere would be represented by:

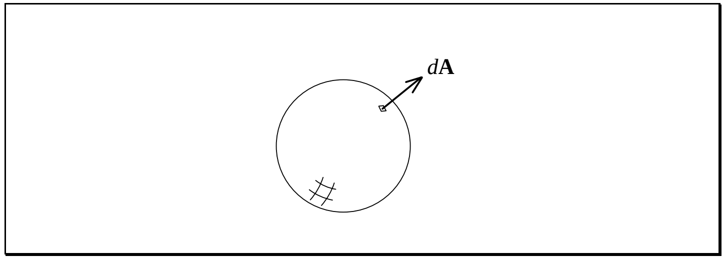


**Figure 1A.5**





## Coulomb's Law

### Demo

*Rub acrylic on rabbits fur. Use long pith ball. Show attraction, then repulsion. Explain in terms of the transfer of "charge". Talk about action-at-a-distance. Rub ebonite. Show attraction and repulsion. Use short pith ball. Show attraction and no repulsion. Postulate the existence of two types of charge. This is exactly what Benjamin Franklin did, and he labelled them as positive and negative.*

*We can "measure" how much charge there is by using an electroscope. Describe the operation of the electroscope. Demonstrate the effect of induction of static charges, and how we can deposit either type of charge on the electroscope – touch the electroscope with the acrylic rod. Touching with ebonite rod should neutralize it. Induce a charge with the ebonite rod, earth electroscope, and remove rod. Introduce ebonite rod.*

### Theory

Coulomb's apparatus

Charles-Augustin de Coulomb won a prize in 1784 by providing the best method of constructing a ship's compass to the French Academy of Sciences. It was while investigating this problem that Coulomb invented his torsion balance. He showed that charge was distributed on the surface of a conductor, and recognized this as a consequence of the mutual repulsion of like charges according to an inverse square law.

In 1785, Coulomb used an apparatus based upon his torsion balance to measure the tiny electrostatic forces caused by two charged spheres. The quantitative results are embodied in Coulomb's Law:

Coulomb's Law

$$\mathbf{F}_1 = \frac{q_1 q_2}{4\pi\varepsilon_0 R^2}\hat{\mathbf{R}} \quad \text{N} \tag{1A.6}$$

The quantities expressed in this law are shown below:

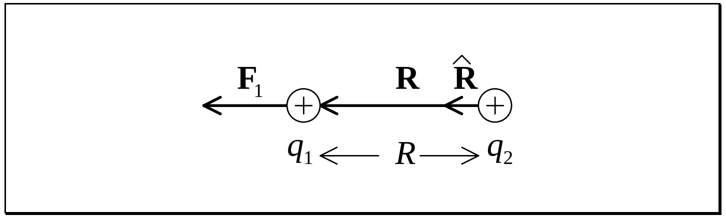


**Figure 1A.6**

The subscript 1 on the force means this is the force on charge $q_1$ due to charge $q_2$. Therefore, the unit vector $\hat{\mathbf{R}}$ points in a direction that goes from $q_2$ to $q_1$.

Mathematical notation

Here we should clarify some notation. The *vector* $\mathbf{R}$ points from the source of the force ($q_2$) to the point where the force is felt ($q_1$). This will *always* be the case throughout this subject. $\mathbf{R}$ points from the source to the effect. The magnitude of the vector $\mathbf{R}$ is just $R$. The unit vector $\hat{\mathbf{R}}$ has the same properties as $\mathbf{R}$ except its magnitude is one.

## The Electric Field

Concept of a "field"

The charge $q_1$ will feel a force even though nothing is touching it! We know that it is caused by $q_2$. We can now imagine some sort of field of influence radiating out from charge $q_2$ into all of space (3 dimensions). As far as $q_1$ is concerned, it just finds itself immersed in some sort of space where a force is felt (think of a rocket in deep space and gravity). We can then imagine that something permeates the space even before we place our charge $q_1$ in it. We





call this a *field*. When we place the charge $q_1$ in the field, we see a reaction – in this case a force.

With this thinking, it appears that a field exists due solely to $q_2$. We call this field the *electric field*, and for a point charge it is defined as:

Electric field of a point charge

$$\mathbf{E}_1 = \frac{q_2}{4\pi\varepsilon_0 R^2}\hat{\mathbf{R}} \quad \text{Vm}^{-1} \tag{1A.7}$$

Electric lines of force

Now consider an isolated point charge. The electric field must exist all around it, but lets examine what happens in a two dimensional cross section. We can take a very small positive test charge and place it in the field near the point charge. We will constrain the test charge to move infinitesimally slowly away from the point charge. The path the test charge traces out is called a line of force.

For an isolated positive point charge, the lines of force radiate from the charge in all directions:

Lines of force give a "picture" of the field

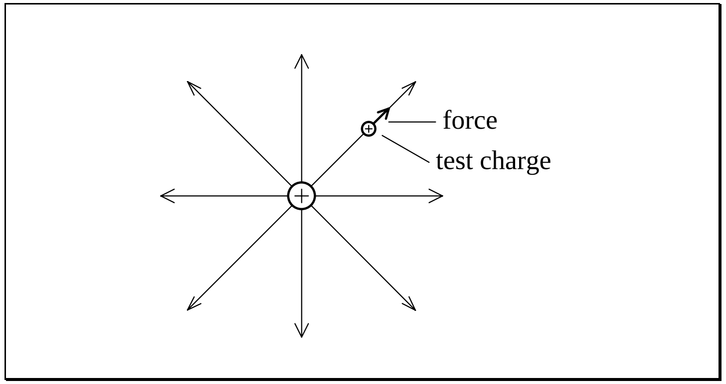


**Figure 1A.7**

The lines of force drawn in this manner create a picture of the electric field. The direction of the electric field at any point is given by the direction of the force on the positive test charge.

Using these ideas, we can calculate the force on a charge using Coulomb's law:

$$\mathbf{F}_1 = q_1\mathbf{E}_1 = q_1 \frac{q_2}{4\pi\varepsilon_0 R^2}\hat{\mathbf{R}} \tag{1A.8}$$

**Computer Demo**

*Demonstrate field around single isolated charge. It is 3D. The direction at any point is given by the tangent to the line of force if more than one charge is involved. Demonstrate with a +ve and -ve charge.*

*Demonstrate the effect of not having a small test charge - it distorts the field.*

**Superposition**

Superposition applied to components of a force

In mechanics we often split up a total force on an object into a number of components. Conversely, we can add up a number of components to get the total force. We can do the same with the electrostatic force. Consider the following arrangement:

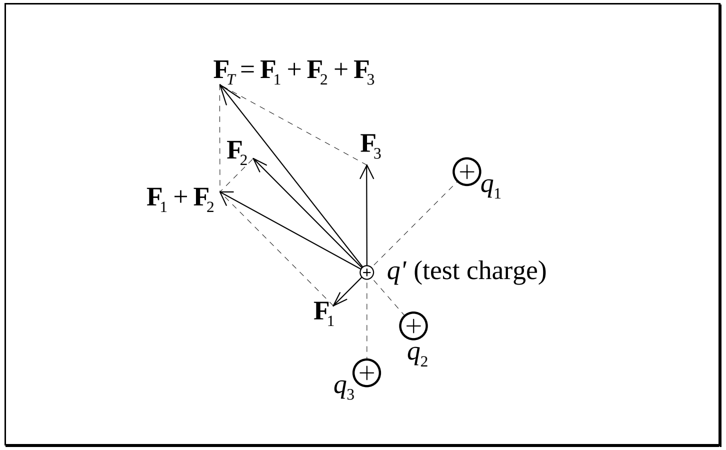


**Figure 1A.8**





The resultant force on the test charge is given by:

Using superposition to calculate the field due to more than one charge

$$\mathbf{F}_T = \sum_{i=1}^{3}\mathbf{F}_i = q'\sum_{i=1}^{3}\mathbf{E}_i = q'\mathbf{E} \tag{1A.9}$$

The constant of proportionality in Coulomb's Law, namely $1/4\pi\varepsilon_0$, is only true for a vacuum. In any other medium, we generalise Coulomb's Law by replacing $\varepsilon_0$ by $\varepsilon$:

Permittivity as an electric property of the medium

$\varepsilon$ = permitttivity (or dielectric constant) of the medium (1A.10a)

$\varepsilon_0$ = permittivity of free space = $8.85419 \times 10^{-12}$ Fm$^{-1}$ (1A.10b)

For media other than free space, we define relative permittivity:

Relative permittivity defined

$$\varepsilon_r = \varepsilon / \varepsilon_0 \tag{1A.11}$$

## Potential Difference

Potential difference defined using mechanical work done

Potential difference involves calculating the work done in moving a test charge between two points. Consider the field from a single isolated charge:

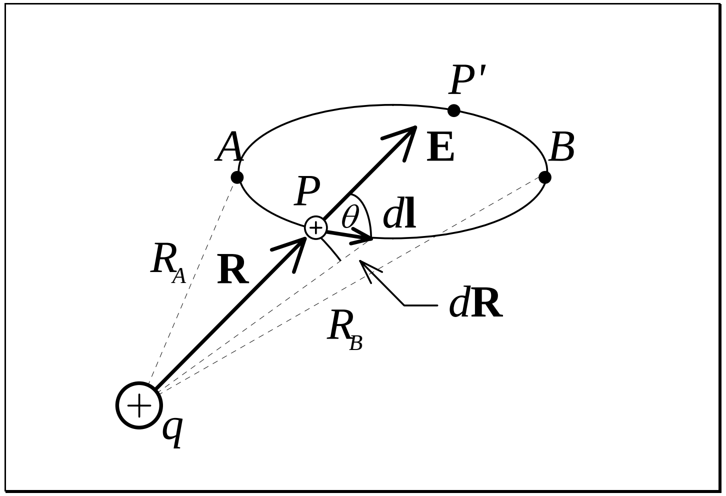


**Figure 1A.9**

The amount of work we have to do to move a charge by a small increment is given by:

Work done depends on the electric field

$$\begin{aligned} dW &= \mathbf{F}_M \cdot d\mathbf{l} \\ &= -\mathbf{F}_E \cdot d\mathbf{l} \\ &= -q\mathbf{E} \cdot d\mathbf{l} \end{aligned} \tag{1A.12}$$

To move a charge infinitesimally slowly in an electric field, we apply a force that exactly counteracts the Coulomb force. A larger force than the Coulomb force will accelerate the charge. The above equation gives the work done when moving an infinitesimal displacement $d\mathbf{l}$. To find the work done in moving a charge from $A$ to $B$, just integrate:

Work done for an arbitrary path in an electric field

$$W_{BA} = -q\int_A^B \mathbf{E} \cdot d\mathbf{l} \quad \text{J} \tag{1A.13}$$





Potential difference is the mechanical work done per unit charge.

Potential difference defined

$$V_{BA} = -\int_A^B \mathbf{E} \cdot d\mathbf{l} \quad \text{V} \tag{1A.14}$$

For the case of a single isolated charge, we calculate the integral as follows:

$$\begin{aligned} V_{BA} &= -\int_A^B E\cos\theta\, dl = -\int_{R_A}^{R_B} E\, dR \\ &= -\int_{R_A}^{R_B} \frac{q}{4\pi\varepsilon R^2}\, dR = \frac{q}{4\pi\varepsilon}\left(\frac{1}{R_B} - \frac{1}{R_A}\right) \end{aligned} \tag{1A.15}$$

For an isolated point charge, we define "absolute potential" at any point to be:

Absolute potential defined for a point charge

$$V = \frac{q}{4\pi\varepsilon R} \tag{1A.16}$$

To find the potential difference at two points, we can subtract the absolute potential of one point with the other. *Compare this with Eq. (1A.15).*

The electrostatic **E** field is a conservative field. This means that no work is done in moving a charge around a path and back to its starting position – energy is conserved. In Figure 1A.9, along *APBP'A* we have:

The electrostatic **E** field is conservative

$$\oint_l \mathbf{E} \cdot d\mathbf{l} = 0 \tag{1A.17}$$

## Current Density and Ohm's Law

Consider a conducting sheet, conductivity $\sigma$, resistivity $\rho$, cross-sectional area $A$:

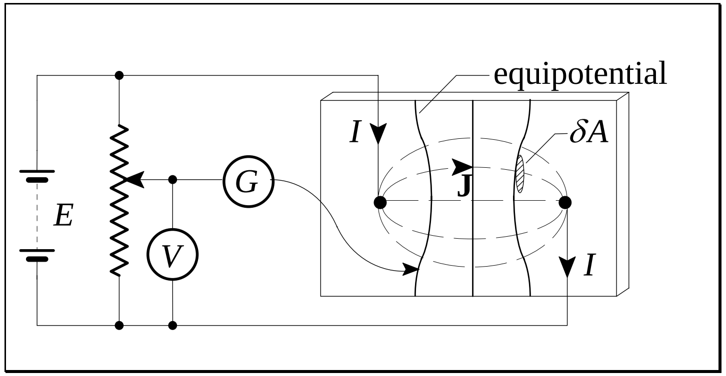


**Figure 1A.10**

The battery $E$ sets up an **E** field, which in turn causes the free charges in the metal sheet to flow along the lines of **E** (lines of force). The current density in the metal sheet is defined as:

Current density defined

$$\mathbf{J} = \lim_{\delta A \to 0} \frac{\delta I}{\delta A} = \frac{dI}{dA} \text{ and tangent to the } \mathbf{E} \text{ lines} \quad (1A.18)$$

Since **J** and **E** point in the same direction, they must differ in magnitude only:

Field version of Ohm's Law

$$\mathbf{J} = \sigma \mathbf{E} \quad (1A.19)$$

*Show that this is Ohm's Law.*

The equipotentials on the metal sheet are determined using the galvanometer $G$. *How?* They are always perpendicular to the lines of **J**.





## Surface Integrals

Consider a uniform current in a conductor. **J** and **A** are both vectors, and:

Getting current from a uniform current density

$$I = \mathbf{J} \cdot \mathbf{A} \quad (1A.20)$$

Does **A** have to be the cross section of the conductor for this expression to be true? *Test this out by drawing an area that is not perpendicular to the conductor cross section.*

What do we do if the current density is not uniform? We divide the area up into regions where the current density is uniform and summate over the whole area. Eventually we come to the surface integral:

Getting current from a current density

$$I = \int_A \mathbf{J} \cdot d\mathbf{A} \quad (1A.21)$$

## Flux and Flux Density

Flux as a tool for describing action-at-a-distance

The action-at-a-distance that we see with electrostatics can be explained by postulating a *flux*, $\psi$, that exerts influence over objects nearby. It does not flow, but emanates, or *streams*, from the source (an electric charge). It permeates all of space. How do we measure the flux that is *streaming* through space?

Think back to how we measured the current going through a conductor. There we defined a current density at each point in the conductor. For electrostatics, if we define an electric flux density at each point in space, and call it **D**, then:

Electric flux density defined

$$\mathbf{D} = \lim_{\delta A \to 0} \frac{\delta\psi}{\delta A} = \frac{d\psi}{dA} \tag{1A.22}$$

When **D** is not uniform, to find the electric flux $\psi$ streaming through an area *A*, we have to perform the integral:

Getting flux from flux density

$$\psi = \int_A \mathbf{D} \cdot d\mathbf{A} \tag{1A.23}$$

*Show that this integral has the same value in free space for all surfaces* A *having the same perimeter*.

Lets make the area a closed area and surround some charge. This gives the closed surface integral:

Getting the flux streaming out of a closed surface

$$\boxed{\psi = \oint_A \mathbf{D} \cdot d\mathbf{A}} \tag{1A.24}$$

*Show this pictorially for a single charge and a sphere*. Obviously, if the charges are the source of flux, then we should get more flux if there is more charge – in our model the amount of flux must be proportional to the amount of charge. This leads to Gauss' Law.





## Gauss' Law for Electrostatics

Gauss' Law can be derived from Coulomb's Law, but it is very complicated. It is simpler to give an intuitive definition:

$$\boxed{\psi = \oint_A \mathbf{D} \cdot d\mathbf{A} = q} \tag{1A.25}$$

Gauss' Law postulated

where *q* is the charge enclosed by the area *A*. In words, it says that the total flux streaming through a closed surface is equal to the amount of charge enclosed by that surface. It *does not* say that no flux can stream out of the enclosing surface – it just means that if some does, then it inevitably must stream back into some other part of the surface.

*Apply Gauss' Law to the point charge and show that*:

$$\boxed{\mathbf{D} = \varepsilon \mathbf{E}} \tag{1A.26}$$

Electric flux density is related to electric field intensity

This is true in general and relates electric *flux density* to electric *field intensity*.

## Summary

- The electrostatic force between two infinitesimally small electric charges is given by Coulomb's Law: $\mathbf{F}_1 = \frac{q_1 q_2}{4\pi\varepsilon_0 R^2}\hat{\mathbf{R}}$ N .
- An infinitesimally small electric charge produces an electric field given by: $\mathbf{E}_1 = \frac{q_2}{4\pi\varepsilon_0 R^2}\hat{\mathbf{R}}$ $\text{Vm}^{-1}$.
- Permittivity, $\varepsilon$ , is an electric property of a medium.
- The potential difference between two points A and B is given by: $V_{BA} = -\int_A^B \mathbf{E}\cdot d\mathbf{l}$ V .
- The electrostatic **E** field is conservative: $\oint_l \mathbf{E}\cdot d\mathbf{l} = 0$ .
- Current density is defined as: $\mathbf{J} = \frac{dI}{dA}$ .
- Ohm's Law is given by: $\mathbf{J} = \sigma\mathbf{E}$ .
- Electric flux density is defined as: $\mathbf{D} = \frac{d\psi}{dA}$ .
- Gauss' Law for electrostatics is: $\oint_A \mathbf{D}\cdot d\mathbf{A} = q$ .
- The electric flux density is related to the electric field intensity by: $\mathbf{D} = \varepsilon\mathbf{E}$.





## References

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Shamos, Morris H. (Ed.): *Great Experiments in Physics - Firsthand Accounts from Galileo to Einstein*, Dover Publications, Inc., New York, 1959.

## Problems

**1.**

Use Gauss' Law to obtain the electrostatic flux density **D** and hence the field intensity **E**, at a distance *r*, in a vacuum, from:

(a) the centre of a uniformly charged spherical shell, with radius *a*, and a total charge *q*, when $r \geq a$.

(b) as (a) but with $r < a$.

(c) a line charge with uniform charge density $\lambda$ $Cm^{-1}$.

(d) a plane with uniform charge density $\sigma$ $Cm^{-2}$.

**2.**

Use the above results to derive expressions for the potential difference between two points at radial distances $r_a$ and $r_b$, if $r_a > r_b$. Draw the field pattern for each case, i.e. lines of force and equipotentials.

**3.**

Derive an expression for the capacitance per unit length of a coaxial cable. The diameters of the inner and outer conductors are $d_1$ and $d_2$ respectively. The insulating material between the conductors has relative permittivity $\varepsilon_r$.

**4.**

A spherical cloud of charge of radius *R* carries total charge *Q*. The charge is distributed so that its density is spherically symmetric, i.e. it is a function of the radial distance from the centre of the sphere.

Explain why the "charge cloud" is equivalent to a point charge of *Q* Coulombs at the centre of the sphere.

Determine the force experienced by an electron, charge -*e*, orbiting the sphere at distance *d* m ( $d > R$ ) from its centre, with constant velocity *v*.

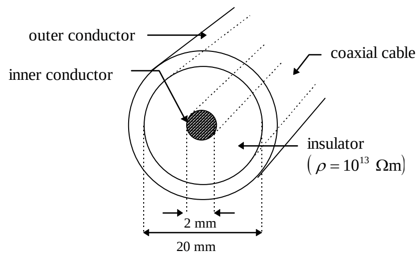




**5.**

Explain the four diagrams of "Faraday's ice pail experiment".

**6.**




The voltage across the insulation layer is 100 kV. Determine the leakage current, for 1 km of cable length, flowing from the inner to the outer conductor.

**7.**

An earthing electrode consists of a metal hemisphere (radius *a* and zero resistivity) just below the surface of the earth (radius ∞, resistivity $\rho$).

Let *a* = 0.5 m, $\rho = 100\ \Omega m$, fault current through electrode = 1 kA.

(a) Show that the resistance to earth (at ∞) is $R = \rho / 2\pi a$.

(b) Determine the resistance between two such electrodes very far apart (i.e. first electrode to ∞ then to second electrode).

(c) Calculate the maximum potential difference between two probes driven into the ground, 0.5 m apart, when the mean distance between the probes and electrode is 100 m.

**8.**

A straight rod $AB$ lies along the $x$-axis and has a uniformly distributed charge density $\lambda$ Cm$^{-1}$.

Show that the $x$ and $y$ components of the **E** field at point $P$ are given by:

$$\mathbf{E}_{Px} = \frac{\lambda}{4\pi\varepsilon_0 b}\left(\sin\theta_B - \sin\theta_A\right)\hat{\mathbf{x}}$$

$$\mathbf{E}_{Py} = \frac{\lambda}{4\pi\varepsilon_0 b}\left(\cos\theta_A - \cos\theta_B\right)\hat{\mathbf{y}}$$

where: $P$ is a point in the first quadrant, $b$ = distance of $P$ from $x$-axis.

$\theta_A$ & $\theta_B$ are the angles $AP$ and $BP$ make with the $x$-axis.

Also show that for a semi-∞ line charge ($A$ at origin, $B$ at ∞, $\theta_A = \pi/2$):

$$E_{Py} = -E_{Px} = \frac{q}{4\pi\varepsilon_0 b}$$

$$\mathbf{E}_P = \frac{q}{2\sqrt{2}\pi\varepsilon_0 b}\angle 135°$$

and for an ∞ line ($A$ at -∞, $B$ at ∞):

$$\mathbf{E}_{Py} = \mathbf{E}_P = \frac{\lambda}{2\pi\varepsilon_0 b}\hat{\mathbf{y}}$$

$$\mathbf{E}_{Px} = \mathbf{0}, \text{ field cylindrical}$$

**9.**

Derive an expression for the capacitance between two spherical, concentric, metal electrodes (radii $R_1$ and $R_2$). The dielectric medium is air.

# Lecture 1B – Electrodynamics

*Magnets. Gauss' Law for magnetostatics. Law of Biot-Savart. Magnetic field near a long, thin conductor. Magnetic scalar potential. Ampère's Law. Reluctance, magnetomotive force and magnetic circuits. The axial field of a current loop. The axial field of a solenoid. The Lorentz force. Electromotive force (emf). Flux linkage. Faraday's Law. Inductance.*

## Revision

### Coulomb's law

Experimental law. Applies to electrostatics.

### Potential Difference

Defined as work done per unit charge in moving charge through an electric field. Equipotentials join all points in a field that have equal potential. The static electric field is a conservative field – no net work is done if you push a charge around some path and finish back at its starting point. *Why?* This is analogous with a gravitational field (on Earth we would have to disregard the friction caused by the atmosphere).

Conservative electric field

### Flux and Flux Density

Flux streams from the source to permeate all of space – it is an imaginary concept. Its message is: "You are now in a force field". It explains how we can have an action-at-a-distance. How does a charge know where another charge is? Flux provides this information. Imagine a picture of the flux. It looks like a picture using lines of force. We say that a tube of flux is the flux in between these lines. *Draw a picture of a charge free region of space with an electric field. (Use lines of force to represent a field). Draw a tube of flux and perform Gauss' Law at each end.*

Flux "explains" action-at-a-distance

Flux tube defined

### Gauss' Law

Can be derived from experimental laws. Easy to apply to cases where geometry has symmetry. Relates flux density to field strength.

## Magnets

A brief history of magnetostatics

In 1600, William Gilbert (physician to England's Queen Elizabeth I) investigated the attraction between magnets and the electrostatic effects observed when certain objects were rubbed. His work, *De Magnete*, was mainly qualitative and provided little understanding of the nature of the phenomena.

Magnetism is similar in some ways (but not others) to electrostatics

The study of magnetism began as the study of mechanical attraction of some objects to certain other objects – not too much different to that of electrostatics. Coulomb showed that there was an inverse square law that applied to the force of attraction between two magnets, like electrostatics. The analogy stops there, because there has been no demonstration of the existence of isolated magnetic poles. *Consider what happens when we try to cut a magnet in half to isolate the two poles.*

### Demo

*Two magnets. Show attraction and repulsion. Show that steel is attracted. Show that it behaves just like an extension of the magnets. Show attraction and repulsion. Remove one piece of steel. Explain? Use one magnet. Sprinkle iron filings. Use compass needle. Filings align along the lines of force, and give a picture of the magnetic field.*





## Gauss' Law for Magnetostatics

An analogous consideration applies to the magnetic field, but in this case there are no isolated sources of the magnetic field so we get:

$$\oint_A \mathbf{B} \cdot d\mathbf{A} = 0 \qquad (1B.1)$$

Gauss' Law applied to magnetics

and:

$$\mathbf{B} = \mu \mathbf{H} \qquad (1B.2)$$

Magnetic flux density is related to magnetic field intensity

The magnetic constant of the medium is the permeability:

$$\mu = \text{permeability of the medium} \qquad (1B.3a)$$

$$\mu_0 = \text{permeability of free space} = 4\pi \times 10^{-7}\ \text{Hm}^{-1} \qquad (1B.3b)$$

Permeability as a magnetic property of the medium

and **H** is the magnetic field strength. We also similarly define:

$$\mu_r = \mu / \mu_0 \qquad (1B.4)$$

Relative permeability defined

## Law of Biot-Savart

Magnetic force is caused by moving charge

In 1819, Hans Christian Oersted found that a wire carrying electric current produces a force similar to the magnetic force.

**Demo**

*Set up the CT and conductor. Show how a horseshoe steel piece is attracted. Sprinkle iron filings to show that the magnetic field is circular - concentric circles of flux. Demonstrate shielding with pieces of steel and aluminium.*

The magnetic field can be produced by (*is?*) a moving electric field.

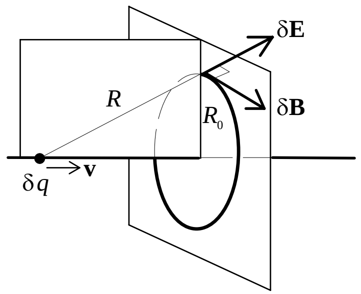



**Figure 1B.1**

With the situation shown above, experiment shows that the magnetic flux density is:

Experimental law relating **B** and **E**

$$\delta\mathbf{B} = \frac{1}{c^2}\left(\mathbf{v}\times\delta\mathbf{E}\right) \tag{1B.5}$$

That is, **B** is perpendicular to both **v** and **E**. In free space, the speed of light is:

$$c = 2.99793\times 10^{8}\ \text{ms}^{-1} \tag{1B.6}$$





But, when considering the charge $\delta q$, we also have:

$$\mathbf{v} = \frac{\delta\mathbf{l}}{\delta t}, \quad \delta\mathbf{E} = \frac{\delta q}{4\pi\varepsilon_0 R^2}\hat{\mathbf{R}},$$

$$i = \frac{\delta q}{\delta t}, \quad \mathbf{B} = \mu_0\mathbf{H} \tag{1B.7}$$

so that the magnetic field strength due to a *current element* $i\delta\mathbf{l}$ in free space is:

Current element defined

$$\delta\mathbf{H} = \frac{1}{\mu_0 c^2}\left(\frac{\delta\mathbf{l}}{\delta t}\right)\times\left(\frac{i\delta t}{4\pi\varepsilon_0 R^2}\hat{\mathbf{R}}\right) \tag{1B.8}$$

In free space, it turns out that:

The relationship between electric and magnetic constants

$$\mu_0 = \frac{1}{\varepsilon_0 c^2} \tag{1B.9}$$

which gives us:

The magnetic field expressed in terms of current

$$\delta\mathbf{H} = \frac{i}{4\pi R^2}\delta\mathbf{l}\times\hat{\mathbf{R}} \tag{1B.10}$$

But current only exists in a closed circuit, so the total magnetic field strength is obtained by adding up all the small contributions of each current element in the circuit $C$:

The Law of Biot-Savart

$$\mathbf{H} = \oint_C \frac{i}{4\pi R^2}\left(d\mathbf{l}\times\hat{\mathbf{R}}\right) \tag{1B.11}$$

This is the Law of Biot-Savart.

## Magnetic Field Near a Long, Thin Conductor

We can apply the Law of Biot-Savart to various conductor arrangements to obtain simple expressions for the magnetic field.

**H** due to a current in a length of long, thin conductor *XY* can be determined this way:

The Law of Biot-Savart applied to a long, thin conductor

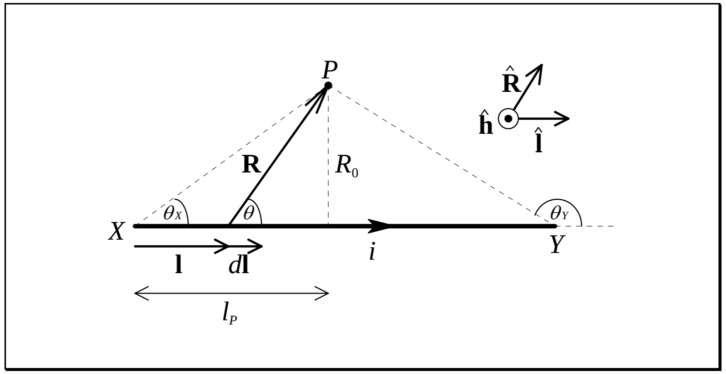



**Figure 1B.2**

Reducing the integrand to a single variable

$$\mathbf{H}_P = \int_X^Y \frac{i}{4\pi R^2}\left(d\mathbf{l}\times\hat{\mathbf{R}}\right)$$

$$l_P - l = R_0 \cot\theta \qquad \frac{dl}{d\theta} = R_0 \operatorname{cosec}^2\theta$$

$$R = R_0 \operatorname{cosec}\theta \qquad d\mathbf{l}\times\hat{\mathbf{R}} = dl\sin\theta\hat{\mathbf{h}}$$

**H** at a point near a long, thin conductor

$$\begin{aligned} H_P &= \int_X^Y \frac{i}{4\pi R^2}\sin\theta dl \\ &= \frac{iR_0}{4\pi R_0^2}\int_{\theta_X}^{\theta_Y}\frac{\operatorname{cosec}^2\theta}{\operatorname{cosec}^2\theta}\sin\theta d\theta \\ &= \frac{i}{4\pi R_0}\left(\cos\theta_X - \cos\theta_Y\right) \end{aligned} \qquad (1B.12)$$

If $XY = \infty$ then $\theta_X = 0,\ \theta_Y = \pi$ and:

**H** at a point near an infinitely long, thin conductor

$$\mathbf{H}_P = \frac{i}{2\pi R_0}\hat{\mathbf{h}} \qquad (1B.13)$$





## Magnetic Scalar Potential

To further draw the analogy between the electric and magnetic field, we might suppose there exists a magnetic potential similar to *V* in electrostatics. Therefore, we define the *magnetic scalar potential* to be:

Magnetic scalar potential defined

$$U_{BA} = -\int_A^B \mathbf{H}\cdot d\mathbf{l} \quad \text{A} \qquad (1B.14)$$

So far, this definition is based upon an analogy with the electric field – it is a purely mathematical relation that requires a physical interpretation.

## Ampère's Law

Consider an infinitely long conductor:

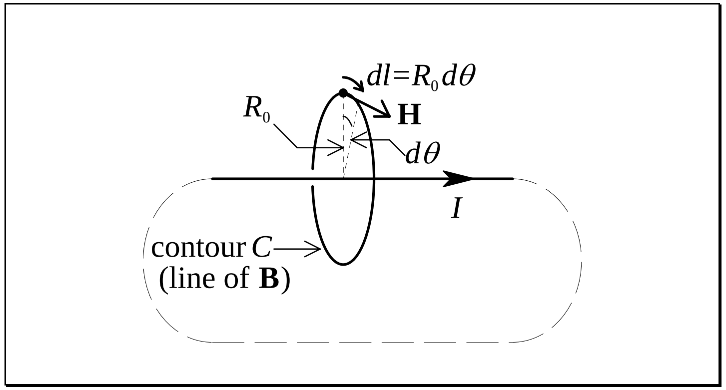



**Figure 1B.3**

If we add up the magnetic scalar potentials around an arbitrary path (contour) that surrounds the current, what do we get? To make the integration in Eq. (1B.14) easier, we choose the arbitrary path to be one that follows a line of **H** so that the dot product is easy to evaluate. *What does the dot product come to in this case?* To get the value of H at each point along the contour, we use Eq. (1B.13) which was derived using the Law of Biot-Savart.

Ampère's Law can be derived from the Law of Biot-Savart

The sum of the magnetic scalar potentials is then:

$$\sum U = \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_0^{2\pi} \frac{i}{2\pi R_0} R_0 d\theta = i \qquad \text{(1B.15)}$$

It turns out that this is the answer we always get, no matter which path $C$ we take. It also turns out that this equation is true for other conductor arrangements. It is so general that we call it Ampère's Law:

Ampère's Law defined

$$\boxed{\oint_C \mathbf{H} \cdot d\mathbf{l} = i} \qquad \text{(1B.16)}$$

*Describe Ampère's Law in words.*

Ampère's Law always applies

Ampère's Law is easy to apply to cases where geometry has symmetry. "Ampère's Law always applies, but *you* may not always be able to apply it." *Demonstrate ease of field calculation with an infinitely long, straight conductor.*

## Reliance, Magnetomotive Force and Magnetic Circuits

**Demo**

*Use a solenoid. Pour iron filings. Equivalent to a magnet. Insert steel rod. Even better. Field looks uniform inside. Maybe if it was infinite? Infinite solenoid = toroid. Pour iron filings. Use solenoid with steel core. Build magnetic circuit. We can direct the flux – like directing current in a wire.*

Making an analogy between electric and magnetic circuits

We can use material with a high permeability to direct the magnetic flux along a certain path. This is analogous to using a wire to conduct a current. By experiment, we have seen that the flux density appears uniform inside air cored solenoids and toroids – maybe it is uniform inside other material as well. Even if the flux density isn't uniform, this may be a good approximation to make.





Magnetic circuits are used where the flux path is well defined

Imagine a *magnetic circuit* that is made up of a high permeability material (such as iron), an air gap, and a winding. For such a circuit, the flux path is well defined, and Ampère's Law reduces to a simple form:

$$i = \oint_C \mathbf{H} \cdot d\mathbf{l} = \oint_C \frac{B}{\mu} dl = \frac{B}{\mu} l = \frac{l}{\mu A} \phi = \mathsf{R}\,\phi \qquad \text{(1B.17)}$$

We call $\mathsf{R}$ the *reluctance* of the magnetic circuit (similar to the concept of resistance in an electrical circuit):

$$\mathsf{R} = \frac{l}{\mu A} \qquad \text{(1B.18)}$$

Reluctance defined

To make the analogy with electric circuits complete, we introduce the *magnetomotive force* (or mmf for short):

$$\mathsf{F} = Ni \qquad \text{(1B.19)}$$

Magnetomotive force (mmf) defined

where $N$ is the number of conductors that carry the current $i$ (e.g. a solenoid). Now Ampère's Law reduces to something similar to Ohm's Law:

Ampère's Law for magnetic circuits

$$\boxed{\mathsf{F} = \mathsf{R}\,\phi} \qquad \text{(1B.20)}$$

## The Axial Field of a Current Loop

We are now in a position to develop some useful formula for the magnetic field intensity caused by some current arrangements. These will be used later on. Consider first a simple circular current loop:

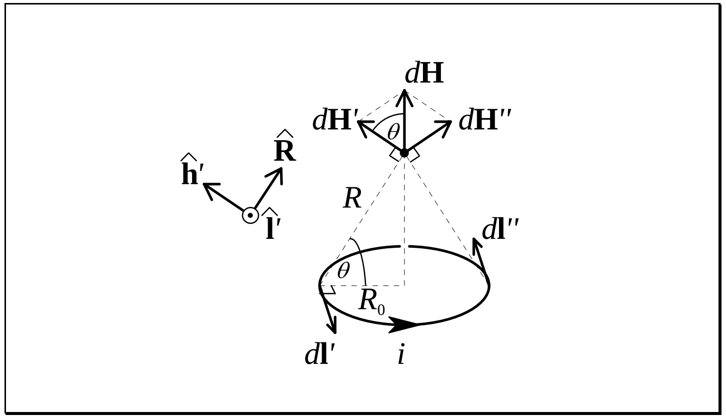



**Figure 1B.4**

We want to find the field at any point on the axis. As can be seen by the figure, we can use symmetry to give the resultant field due to two opposite current elements:

$$dH = 2dH'\cos\theta \qquad \text{(1B.21)}$$

By the Law of Biot-Savart, and noting that the angle between $d\mathbf{l}'$ and $\hat{\mathbf{R}}$ is always 90°, we have:

$$dH = \frac{2i}{4\pi R^2} dl' \cos\theta \qquad \text{(1B.22)}$$





Adding up all the contributions of the current elements as we go around current loop, we get:

$$H = \int_0^{\pi R_0} \frac{2i\cos\theta}{4\pi R^2} dl' = \left[ \frac{i(R_0/R)l'}{2\pi R^2} \right]_0^{\pi R_0}$$

$$\mathbf{H} = \frac{iR_0^2}{2R^3}\hat{\mathbf{h}} = \frac{iA}{2\pi R^3}\hat{\mathbf{h}} \qquad \text{(1B.23)}$$

## The Axial Field of a Solenoid

A solenoid can be approximated by a number of current loops

We can immediately use the previous result to derive the axial field of a circular solenoid. Let the solenoid have $N$ turns, axial length $l$, radius $R_0$, diameter $d = 2R_0$ and current $I$. To approximate the windings of the solenoid (which form a spiral), we can say that since they are so close together, they look like many independent loops.

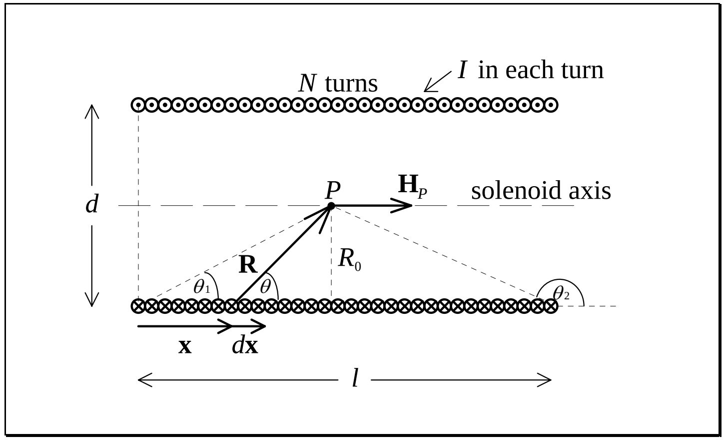



**Figure 1B.5**

In length $dx$ there are $N/l\,dx$ turns.

We first find the contribution to the total field that is due to one of the current loops:

$$x_P - x = R_0 \cot\theta \qquad \frac{dx}{d\theta} = R_0 \operatorname{cosec}^2\theta \qquad R = R_0 \operatorname{cosec}\theta \tag{1B.24}$$

Application of Eq. (1B.23) gives:

$$d\mathbf{H} = \frac{Ni}{l} dx \frac{R_0^{\,2}}{2R^3}\hat{\mathbf{h}} = \frac{Ni}{2l}\sin\theta d\theta\,\hat{\mathbf{h}}$$

**H** on the axis of a solenoid

$$\mathbf{H}_P = \frac{Ni}{2l}\int_{\theta_1}^{\theta_2}\sin\theta d\theta\,\hat{\mathbf{h}} = \frac{Ni}{2l}\left(\cos\theta_1 - \cos\theta_2\right)\hat{\mathbf{h}} \tag{1B.25}$$

Since:

$$\cos\theta_1 = \frac{l}{\sqrt{d^2+l^2}}, \quad \cos\theta_2 = \frac{-l}{\sqrt{d^2+l^2}} \tag{1B.26}$$

then:

**H** on the axis of a real solenoid

$$\mathbf{H}_P = \frac{Ni}{\sqrt{d^2+l^2}}\hat{\mathbf{h}} \tag{1B.27}$$

If $l >> 2R_0$ (or $\cos\theta_1 \approx 1,\ \cos\theta_2 \approx -1$), then:

**H** on the axis of an infinite solenoid

$$\mathbf{H}_P = \frac{Ni}{l}\hat{\mathbf{h}} \tag{1B.28}$$





## The Lorentz Force

**Demo**

*The turns of a "loose" solenoid experience attraction when carrying a current. Two parallel conductors experience attraction when carrying current in the same direction, repulsion when currents are in opposite direction.*

A moving charge will interact with **B**

A charge moving through a magnetic field (caused by, for example, a current or a permanent magnet) will experience a force.

Consider the case of the two wires attracting each other. The wires need not be parallel to experience a force:

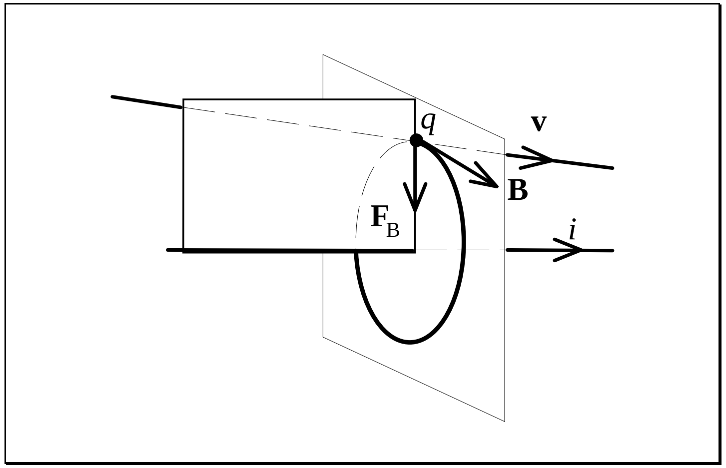



**Figure 1B.6**

From experiments, the magnetic force is given by:

The force on a charge moving through **B**

$$\mathbf{F}_B = q\left(\mathbf{v}\times\mathbf{B}\right) \tag{1B.29}$$

The total (electric and magnetic) force experienced by the charge $q$ is given by the Lorentz Force Law:

Lorentz Force Law

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right) \tag{1B.30}$$

## Electromotive Force (emf)

Consider a conductor (with lots of free electrons) moving in a magnetic field.

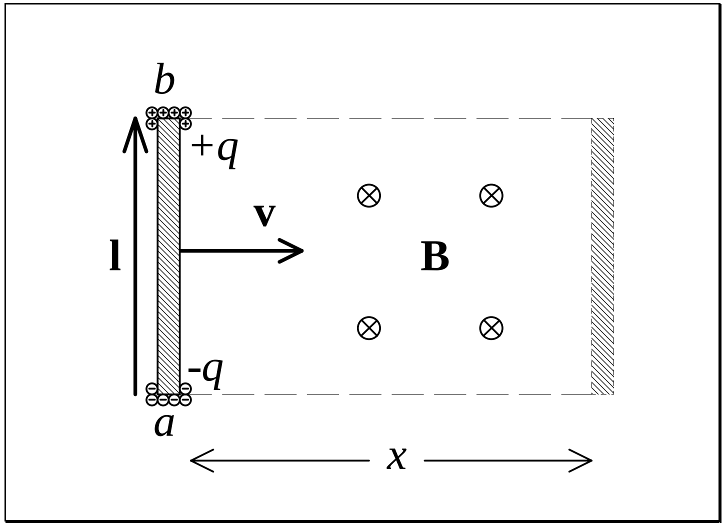



**Figure 1B.7**

A conductor moving in a magnetic field has a voltage impressed across it

The Lorentz force applies and acts on each electron. The electrons experience a force, and will move in the direction of the force. *Show that the electrons experience a force downwards*. Charge separation results. An electrostatic force (a Coulomb force) now also exists between the charges at the ends of the rod. An *equilibrium* or *steady state* will be reached when:

$$\mathbf{F}_E + \mathbf{F}_B = \mathbf{0} \tag{1B.31}$$

When this occurs, the electrons will cease moving, since there is no force on them.

At equilibrium, the voltage can be deduced using electrostatics

Since we have a static situation in the vertical direction as far as the charges are concerned (they are not moving vertically), we can use our electrostatic knowledge to calculate the work per unit charge involved in moving a charge along the conductor.





Define electromotive force (emf), or more properly, voltage:

The voltage developed across a straight conductor in a uniform field

$$V_{ba} = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = \int_a^b (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = vBl \tag{1B.32}$$

Voltage and potential difference, although similar concepts, are different things. Voltage can cause a steady current, potential difference cannot.

Voltage is different to potential difference

For example, suppose we have a capacitor that has charge on its plates, and hence a potential difference between them. If we connect a resistor across the plates, the charge will flow through the resistor and the capacitor will discharge. There is no mechanism to restore charge on the capacitor, so the current will be short-lived and decay to zero.

A voltage requires an energy source

With the moving conductor, we can connect a resistor across ends *a* and *b*, and charge will flow through the resistor. As soon as charge escapes from the ends in this manner, Eq. (1B.31) does not apply, and electrons will feel a net force again. Thus, there is a mechanism operating that restores the lost charge. This arrangement will therefore support a steady current. The moving conductor is like a pump – it can push electrons through a resistor.

This is the principle of the generator.

A current in a magnetic field experiences a force

Now consider the same arrangement, but used in a different way. If there is a current in the conductor which is immersed in the magnetic field, then the Lorentz force law applies in a different way. The force is on the moving charges which flow through the conductor:

which is just another form of the Lorentz Force Law

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}) = it\left(\frac{1}{t}\mathbf{l} \times \mathbf{B}\right) = i(\mathbf{l} \times \mathbf{B}) \tag{1B.33}$$

The force on the charges is translated to a force on the rod, causing it to move.

This is the principle of the motor.

## Flux Linkage

Consider a loop of wire immersed in a magnetic field:

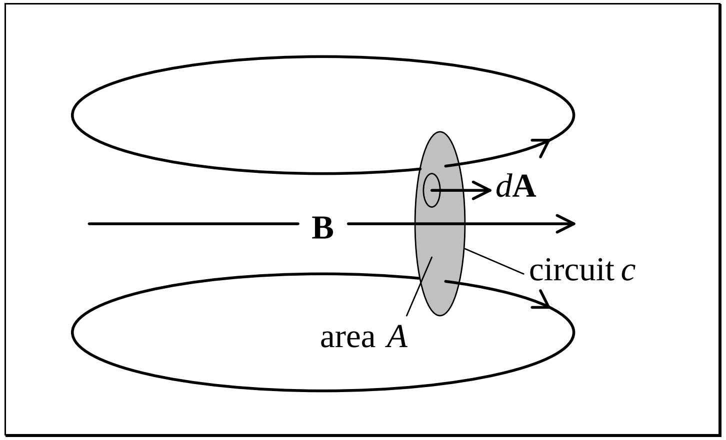



**Figure 1B.8**

Flux linkage explained,

A flux line (which is a closed loop) that passes through the circuit (which is a closed loop), forms a "link", like two links in a chain. To calculate the total amount of flux linking the circuit, we use:

how we calculate it,

$$\phi = \int_A \mathbf{B} \cdot d\mathbf{A} \tag{1B.34}$$

This is analogous to the calculation of electric flux and current seen previously.

If there are $N$ turns in a solenoid (which is approximately $N$ individual loops), then the flux that links with the circuit is the sum of all the flux linking each turn. We define this flux linkage to be:

and generalised to circuits where flux can link more than once

$$\begin{aligned} \lambda &= \phi_1 + \phi_2 + \phi_3 + \ldots + \phi_N \\ \phi_{av} &= \frac{\phi_1 + \phi_2 + \phi_3 + \ldots + \phi_N}{N} = \frac{\lambda}{N} \\ \lambda &= N\phi_{av} \end{aligned} \tag{1B.35}$$





## Faraday's Law

**Demo**

*Connect a meter to a coil and move a magnet nearby. The meter deflects with each movement of the magnet. It is the Lorentz Force law in action. Spin a magnet from the end of a string to demonstrate a simple generator.*

*We have seen that the magnetic field of a solenoid looks like that of a permanent magnet. We can repeat the experiment with a solenoid. It still works.*

Induced voltage is caused by a rate of change of flux linkage

In 1840, Michael Faraday discovered a simple relationship to describe the phenomenon of induced voltage. When the magnetic flux linking a circuit changes, a voltage is induced in the circuit. Faraday's Law says the induced voltage is equal to the rate of change of magnetic flux:

Faraday's Law

$$e = -\frac{d\lambda}{dt} \tag{1B.36}$$

It can be derived from the Lorentz force, but it was discovered experimentally.

Using partial differentiation and the chain rule, we can also write Faraday's Law as two components – a *transformer* voltage and a *motional* voltage:

Transformer voltage and motional voltage

$$e = -\frac{d\lambda}{dt} = -\frac{\partial\lambda}{\partial t} - \frac{\partial\lambda}{\partial x}\frac{dx}{dt} \tag{1B.37}$$

Lenz's Law

Lenz's Law can be used to give the induced voltage a polarity in a circuit diagram. The polarity of the voltage is such as to oppose the *change* causing it.

Field version of Faraday's Law determines voltage polarity, but Lenz's Law is easier

In reality, Faraday's Law is enough to determine the polarity of the induced voltage, but it depends on the way in which we define the path of integration (refer back to Eq. (1B.32) to see this). For our purposes, Eq. (1B.36), without the minus sign, will give us the *magnitude* of the voltage, and we will use Lenz's Law to give us the *polarity* of the voltage.

An electrical circuit can show magnetic effects - the inductor

If the circuit is closed, then there will be a current. The electrical equivalent circuit of our loop of wire is:

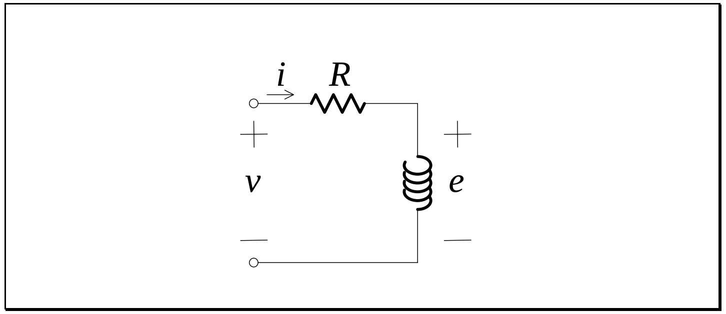



**Figure 1B.9**

In this equivalent circuit, $R$ represents the resistance of the wire, and $e$ is the induced voltage caused by the changing magnetic flux linkage. KVL gives:

$$v = Ri + e \tag{1B.38}$$

## Inductance

### Self Inductance

For linear media, flux linkage is proportional to current

Imagine loops of wire wrapped around a toroid with a core material that has a constant permeability (such as plastic or wood). The amount of flux linking the circuit will then be directly proportional to the current in the loop:

$$\begin{aligned}\lambda &= N\phi = NBA = N\mu HA \\ &= N\mu \frac{Ni}{l} A = \frac{N^2}{l/\mu A} i \\ &= \frac{N^2}{\mathcal{R}} i \\ &= Li\end{aligned} \tag{1B.39}$$





We call the constant of proportionality the inductance. Its value depends only on geometric factors (lengths, areas, etc) and the material ($\mu$), but is normally determined by:

$$L = \frac{\lambda}{i} \tag{1B.40}$$

Inductance defined

*Show the analogy with capacitance.*

### Mutual Inductance

When flux links with another circuit, we need the concept of mutual inductance

We have seen that a changing magnetic field causes an induced voltage. If a coil is producing the magnetic field, as it did above, then the flux linkage depends directly upon the coil's current – this is summarised in Eq. (1B.40). What if the magnetic field linking a circuit is not caused by itself? We then have to define "mutual" inductance.

Consider two coils wound in the same direction:

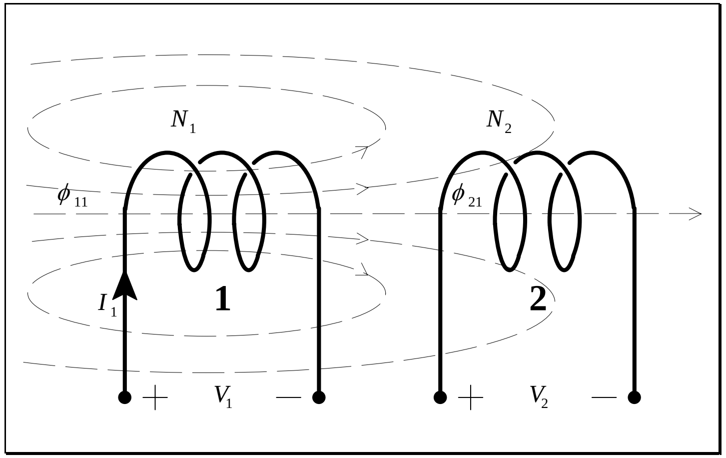



**Figure 1B.10**

Some of the flux produced by Coil 1 links with Coil 2. We define:

Flux linkage is generalised further

$$\lambda_{11} = N_1\phi_{11} = \text{flux linking 1 due to 1} \qquad \text{(1B.41a)}$$

$$\lambda_{21} = N_2\phi_{21} = \text{flux linking 2 due to 1} \qquad \text{(1B.12b)}$$

We can now define mutual inductance:

Mutual inductance defined

$$L_{21} = \frac{\lambda_{21}}{i_1} \qquad \text{(1B.42)}$$

An emf can be caused by another circuit

What if Coil 1 is excited in such a way as to produce a changing magnetic field? (A sinusoidal voltage could do the job.) Coil 2 is an open circuit, but it has been immersed in the field of Coil 1. Magnetic flux produced by Coil 1 will link with Coil 2. An *induced voltage* will be produced across its terminals due to Faraday's Law:

$$v_2 = e_2 = -\frac{d\lambda_{21}}{dt} \qquad \text{(1B.43)}$$

*If the voltage across Coil 2 has the polarity shown in the diagram, is* $\phi_{21}$ *increasing or decreasing?*





## Summary

- Gauss' Law for magnetostatics is: $\oint_A \mathbf{B} \cdot d\mathbf{A} = 0$.
- The magnetic flux density is related to the magnetic field intensity by: $\mathbf{B} = \mu\mathbf{H}$.
- The Law of Biot-Savart relates magnetic field intensity to the current elements that cause it by: $\mathbf{H} = \oint_C \frac{i}{4\pi R^2}\left(d\mathbf{l} \times \hat{\mathbf{R}}\right)$.
- Ampère's Law relates magnetic field intensity around a conductor to the enclosed current by: $\oint_C \mathbf{H} \cdot d\mathbf{l} = i$.
- Ampère's Law for a magnetic circuit states: $\mathcal{F} = \mathcal{R}\phi$.
- A charge moving through electric and magnetic fields will experience a Lorentz force: $\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$.
- A flux tube streaming through a circuit creates a flux linkage which is given by: $\phi = \int_A \mathbf{B} \cdot d\mathbf{A}$.
- When the magnetic flux linking a circuit changes, a voltage is induced in the circuit, given by Faraday's Law: $e = -\frac{d\lambda}{dt}$.
- The amount of self-generated flux linking a circuit is directly proportional to the current in the circuit. The constant of proportionality is called the self inductance, and is given by: $L = \frac{\lambda}{i}$.
- The amount of externally-generated flux linking a circuit is directly proportional to the current that generates the flux. The constant of proportionality is called the mutual inductance, and is given by: $L_{21} = \frac{\lambda_{21}}{i_1}$.
- A changing magnetic flux generated by one circuit can produce a voltage in another. This is called an induced voltage.

**References**

Plonus, Martin A.: *Applied Electromagnetics*, McGraw Hill Kogakusha, Ltd., Singapore, 1978.

Shadowitz, Albert: *The Electromagnetic Field*, Dover Publications, Inc., New York, 1975.

Shamos, Morris H. (Ed.): *Great Experiments in Physics - Firsthand Accounts from Galileo to Einstein*, Dover Publication, Inc., New York, 1959.





## Problems

**1.**

Calculate the field intensity **H** and the flux density **B** at the centre of a current loop, radius 50 mm, carrying current $I = 5$ A when the loop is wound on a core made of:

(a) air

(b) aluminium

(c) iron $\left(\mu_r = 10000\right)$

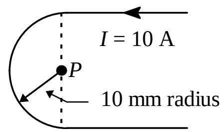
**2.**

A plane square loop of wire, side $l$, carries clockwise current $I$.

Show that at the centre $\mathbf{H} = 2\sqrt{2}I/\pi l$ (down).

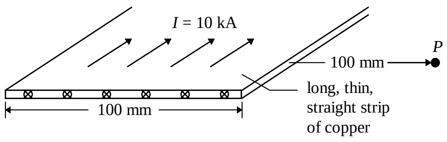
**3.**

Determine the magnetic flux density at point $P$.



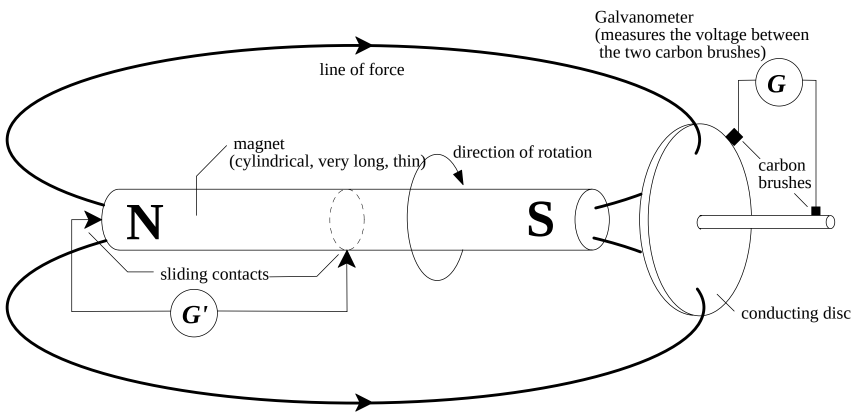

**4. [ Current sheet ]**

Calculate the flux density at a point $P$, 100 mm from an edge, and on the same plane of the strip.




Assume: Current is uniformly distributed.
There are no magnetic materials in the vicinity.

**5. [Faraday's Disc Experiment]**




Is there an emf induced in the galvanometers *G* and *G'*?
Is Lenz's Law obeyed?

In the following cases:

(a) Disc only rotates.

(b) Magnet only rotates.

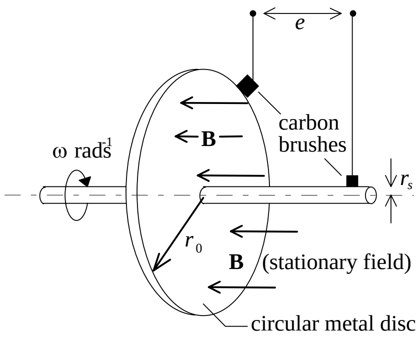
(c) Both rotate in same direction.





**6.**

The uniform magnetic field, of density *B* Tesla is perpendicular to the plane of the disc.




(a) Make a sketch showing the magnetic forces on the free electrons in the disc, the charge distribution on the disc and the electrostatic forces on the free electrons.

(b) Derive an expression for the voltage *e* between the carbon brushes on the outside rim and on the surface of the shaft.

**7.**

A 132 kV, 500 A (RMS), single phase transmission line has two conductors, each 20 mm in diameter and 1.5 m apart. The span between supporting poles is 200 m.

Determine the average force acting on the conductors, over one span, during "short circuit" conditions if the short circuit current = 12 x normal current (ignore line sag).

**8.**

Two parallel circuits of an overhead power line consist of four conductors carried at the corners of a square.

Find the flux per unit length of circuit, in webers/km, linking one of the circuits when there is a current of 1 A in the other circuit.

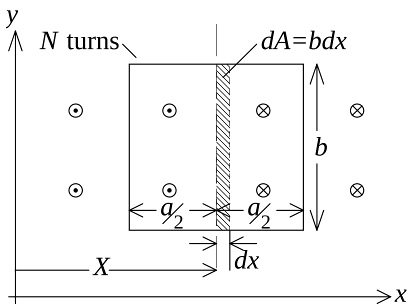
# 1B.26

**9.**

Use the Lorentz Force Law to determine the force experienced by a 10 m long conductor carrying current $I$ = 100 A (S to N) due to the Earth's magnetic field ($B = 5\times 10^{-5}$ T, angle of dip 60°).

**10. [ Coil moving in magnetic field]**

Consider a small rectangular coil ($N$ turns into paper) wound on a non-magnetic former and in a magnetic field of density $B$ (perpendicular to paper).




(a) Calculate the flux linking the coil if $B = \hat{B}\sin\alpha x$ where $\alpha$ = constant.

(NB. Use strip $dA$, $\therefore\ d\phi = BdA$ and $\phi = \int_{X-a/2}^{X+a/2} \hat{B}\sin\alpha x\ bdx$)

(b) Calculate the emf $e$ induced in the coil when it moves in the $x$-direction with speed $v$.

(c) Determine the value of $a$ which maximises $e$ determined in (b).

(d) Calculate the induced emf $e$ if $B = \hat{B}\sin(\omega t - \alpha x)$ and the coil moves in the $x$-direction with velocity $v$ ($\omega$ = constant), and determine $e$ when $v = \omega/\alpha$ .

# Lecture 2A – Conductors and Insulators

*The Bohr atom. The energy barrier. Bound and free charges. Conduction. Dielectrics. The electric dipole. Effect of Boundaries. Polarisation. Ferroelectrics. Breakdown at sharp points.*

## The Bohr Atom

We need to investigate the behaviour of matter subjected to electric fields. This will aid in the understanding of conductors and insulators. We will use the Bohr model of the atom (Greek: *a* = not, *tomos* = cut. The first model of the atom was proposed by Democritus around 430 BCE, about one century prior to the time of Aristotle).

Matter modelled by the atom

John Dalton, in 1803, put forward the idea that the atomic nature of substances could be used to explain their chemical properties. In 1911, Rutherford performed experiments that revealed the structure of atoms. He proposed a model where a very small, positively charged nucleus which carries virtually all the mass of the atom is surrounded by a number of negatively charged electrons.

An atom made up of electric charges

Since there is charge separation, there must be a Coulomb force between the nucleus and electron. To account for the fact that electrons remain at large distances from the nucleus, despite the force of attraction, we assume that the electrons revolve in orbits (like the solar system). Since an electron moving is equivalent to a current, there must be a magnetic field. For the moment, we will consider only the electric field. Magnetic effects resulting from electron motion will be looked at later.

"Solar system" model of the atom

At the atomic level, protons and electrons obey Coulomb's Law, but the internal effects of this are not always detected externally. Consider a solid piece of metal, which is made up of a lattice of nuclei and a "sea" of electrons. The distance between an electron and nucleus is approximately $10^{-10}$ m. Coulomb's Law gives $E \approx 10^{12}\ \text{Vm}^{-1}$. Violent variations in $E$ would be expected along the path of a wandering electron, but average out and large

Microscopic view of matter sees large fields

Macroscopic view of matter does not see large fields

variations are not detected externally. It is clear that we have to delve inside the atom to explain some physical properties that are observed externally.

Bohr's model of the atom

The resulting problems with the Rutherford model (accelerating charge radiates energy – where does this energy come from?) are overcome by using Bohr's model. He proposed that there were certain stable orbits in which electrons could exist indefinitely.

Consider the hydrogen atom, because it is the easiest to analyse:

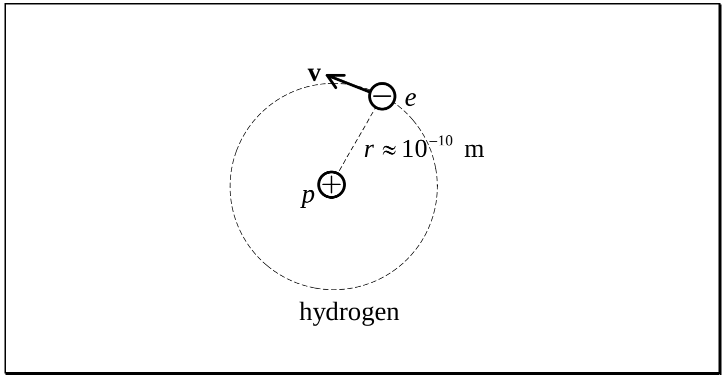



**Figure 2A.1**

Steps leading to the energy of an electron in Bohr's model of the atom

The electrostatic force of attraction between the charges:

$$F = \frac{e^2}{4\pi\varepsilon_0 R^2} \tag{2A.1}$$

provides the centripetal force and, from Newton's second law, since the electron is orbiting with constant speed:

$$\frac{e^2}{4\pi\varepsilon_0 R^2} = \frac{mv^2}{R} \tag{2A.2}$$





Therefore, the kinetic energy of the electron at radius $R$ is:

Kinetic energy of a bound electron

$$K = \frac{1}{2}mv^2 = \frac{e^2}{8\pi\varepsilon_0 R} \tag{2A.3}$$

The potential energy is defined with zero potential at infinity (analogy with gravitational potential energy defined with respect to ground level):

Potential energy of a bound electron

$$\begin{aligned} U &= \int_\infty^R \mathbf{F}_M \cdot d\mathbf{l} \\ &= \int_\infty^R -\mathbf{F}_E \cdot d\mathbf{l} \\ &= \int_\infty^R \frac{e^2}{4\pi\varepsilon_0 r^2} dr \\ &= \frac{-e^2}{4\pi\varepsilon_0 R} \end{aligned} \tag{2A.4}$$

so the total electron energy at radius $R$ is:

Total energy of a bound electron depends on *R*

$$E = U + K = \frac{-e^2}{8\pi\varepsilon_0 R} \tag{2A.5}$$

For a stable orbit, the radius determines the total energy of the electron (electron energy is always negative and more so at smaller orbits).

## The Energy Barrier

Concept of energy barrier is introduced using a uniform field

Consider two equipotentials ($A$ and $B$) in a uniform **E** field:

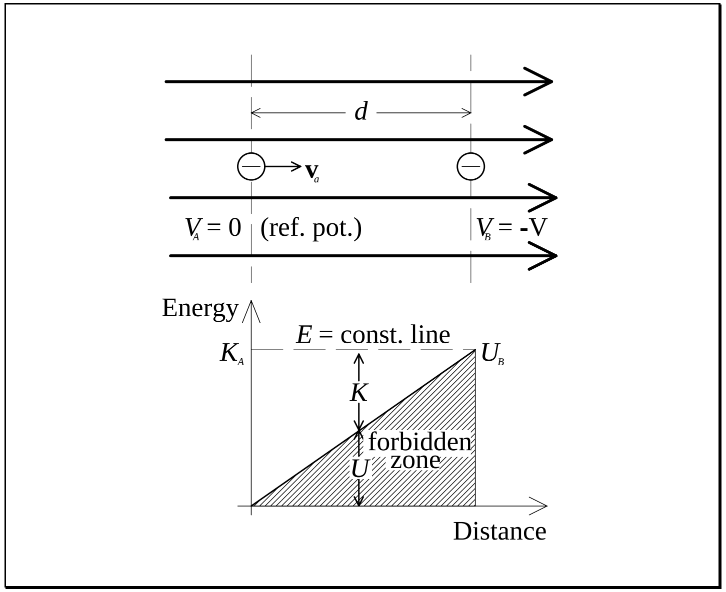



**Figure 2A.2**

As the electric field is conservative (no energy lost in the system), conservation of energy applies and:

The energy of a charge is constant because the electric field is conservative

$$E_A = U_A + K_A = \overset{=0}{\cancel{eV_A}} + \tfrac{1}{2} m_e v_a^2 = \text{const} \tag{2A.6}$$

An energy barrier converts all energy to potential energy - it stops charges from moving

All the energy at $A$ is kinetic energy. As the electron travels towards $B$, some of this kinetic energy is converted into potential energy – the electron must be slowing down. At $B$, the electron has come to rest so $K_B = 0$. At this point, the energy of the electron is all in the form of potential energy. This potential energy has the "potential" to do work, for the electron reverses its motion and returns towards $A$. No matter what the energy is to start with, the electron can never get past the "energy barrier" caused by the electric field, i.e. the shaded area of Figure 2A.2 cannot be entered.





## Bound and Free Charges

We know from the Bohr model that if an electron is in a stable orbit, then its energy is a constant and given by Eq. (2A.5). We can graph the electron energy for this case as a function of distance $x$ from the nucleus:

Bound electron energy graphed as a function of position from the nucleus

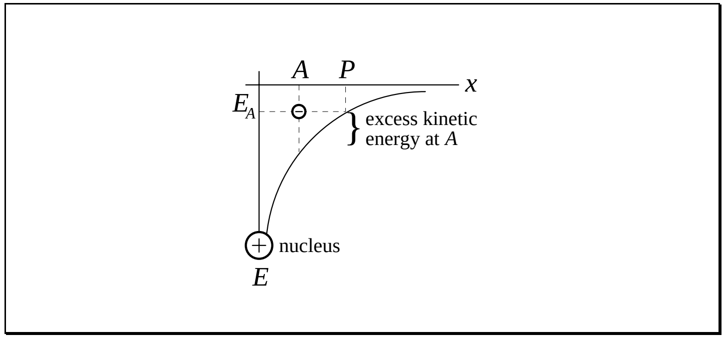



**Figure 2A.3**

The graph shows an energy barrier for electrons - we can determine if an electron is free or bound from it

Let's suppose an electron is at position $A$ with a certain energy $E_A$. Is it in a stable orbit, or is it free? We know where it would be if it were in a stable orbit (right on the line of the graph), and we know that kinetic energy is positive. Therefore, if it lies above the graph, then it is free to move about. It can only move as far as point $P$, where all its energy is used in orbiting a nucleus. *Can a stable position exist below the graph?*

In the presence of two nuclei, the energy of a bound electron looks like:

The energy barrier applies to more than one nucleus

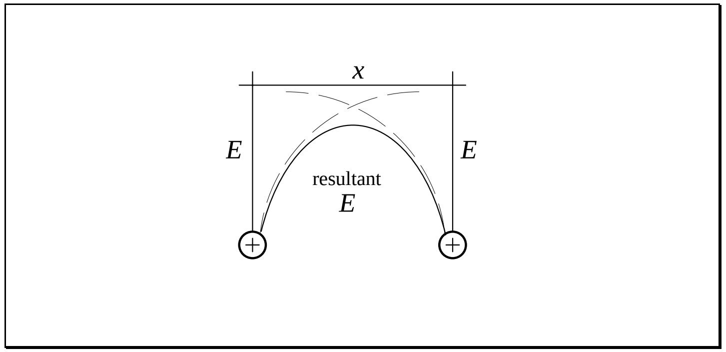



**Figure 2A.4**

The energy barrier is used as a model of matter

If we consider the edge of a material, the energy for a bound electron has the form of:

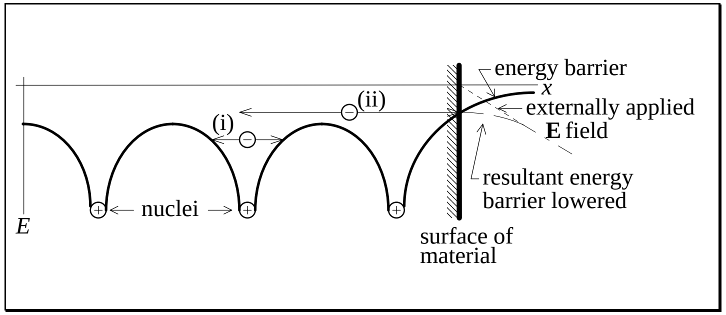



**Figure 2A.5**

This is our model for the inside of a metal. On a microscopic scale, the potential varies greatly near a nucleus. Electrons can be one of two types, depending on their energy:

Bound electrons are trapped by the energy barrier

(i) *Bound charge* – lower energy electron, trapped in a “potential well”. It cannot leave the parent atom (or atoms).

Free electrons have more energy which is used to overcome the energy barrier

(ii) *Free charge* – higher energy electron, above the top of the potential wells. They are free to move in the crystal lattice. They contribute to the current when an external voltage is applied.





**Escape from a Surface**

An energy barrier exists at the surface of a material

If a “free electron” crosses the surface of the material, it comes up against an energy barrier. It “falls” back into the metal, unless the “escape energy” is greater than the “barrier energy”.

The surface energy barrier can be overcome in a number of ways

An electron can escape from the surface in a number of ways:

(i) We can lower the barrier energy by using an externally applied **E** field. (Called the Schottky effect, or high field emission).

(ii) We can increase the escape energy by heating the material. The increased thermal agitation means that some electrons will have enough energy to overcome the barrier.

(iii) Photons can also free electrons (discovered by Einstein) – called the photoelectric effect.

## Conduction

Free electron motion is normally random

The free electrons in a metal drift randomly in all directions and constitute thermal motion. The average drift velocity of all electrons is zero.

Free electrons move in response to an applied field

Free electrons colliding with the lattice produce heat

An applied voltage across a conductor gives rise to an **E** field within the conductor. After electrons have been accelerated by the field, they collide with the lattice, resulting in energy transfer to the lattice in the form of heat. This gives rise to an average drift velocity within the conductor, based upon Newton's second law and the concept of relaxation time (the mean free time between collisions). From this, Ohm's law can be derived.

Ohm's Law can be obtained experimentally

Alternately, we could set up an experiment to measure the resistance of a specimen:

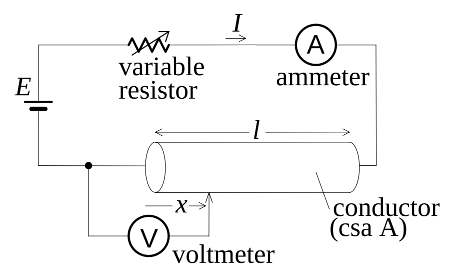



**Figure 2A.6**

If the applied field is varied (by changing the variable resistor), the voltage $V_x$ along the conductor varies smoothly – no violent changes are observed. A linear $V_x$ - $I$ relationship results, with slope $R = V_x / I$ being the conductor resistance.

Resistivity defined

Resistivity $\rho$ is obtained from:

$$R = \frac{\rho l}{A} \tag{2A.7}$$





Resistance varies with temperature

The variation of $R$ with temperature can be investigated by placing a specimen in a calorimeter and repeating the above experiment at various temperatures:

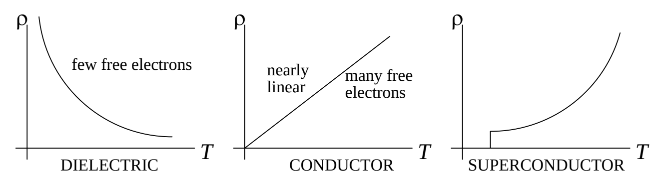



**Figure 2A.7**

and varies with the type of material

For dielectrics (insulators), the resistivity decreases with increasing temperature because electrons are being freed (remember potential wells). For metallic conductors, the increase in resistivity is due to larger thermal vibrations of the lattice atoms. The density of conduction electrons is essentially independent of temperature.

For metal, resistivity varies approximately linearly

For metallic conductors, the relationship between resistivity and temperature is nearly linear, so we can express resistivity as a function of temperature easily:

$$\rho = \rho_0 \left[ 1 + \alpha \left( T - T_0 \right) \right] \tag{2A.8}$$

Temperature coefficient defined

where we have defined the temperature coefficient:

$$\alpha = \frac{1}{\rho} \frac{\delta \rho}{\delta T} \tag{2A.9}$$

## Dielectrics

Dielectrics defined - insulators

Dielectrics are materials that consist of atoms with their outermost electron shell almost completely filled. It is therefore relatively difficult to dislodge an electron from the shell – it is a bound electron. Dielectrics therefore have few electrons available for conduction and are classified as insulators.

## The Electric Dipole

Electric dipole defined

A dipole consists of equal positive and negative charges held a distance apart. For example, the displacement of the electron cloud from the nuclei in an atom (due to an applied field) is a dipole. Another example is polar molecules which have the centres of positive and negative charge permanently displaced due to chemical bonding.

and modelled

The electric dipole can be represented by:

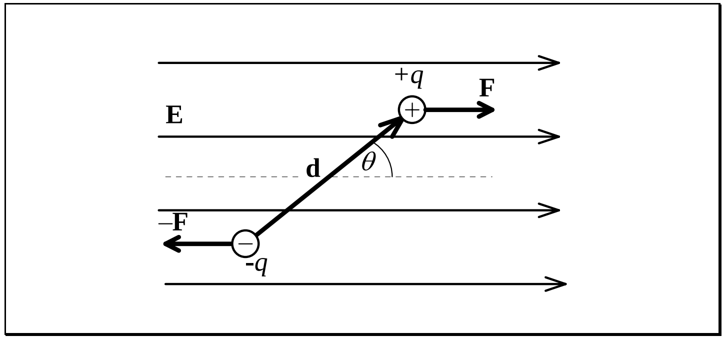



**Figure 2A.8**

A dipole tends to align with the applied **E** field. Coulomb's Law gives:

$$\mathbf{F} = q\mathbf{E} \tag{2A.10}$$





Therefore, the torque experienced by a dipole when it rotates due to an applied field is:

$$T = 2\frac{d}{2}F\sin\theta = qdE\sin\theta \tag{2A.11}$$

We define the electric dipole moment:

$$\boxed{\mathbf{p} = q\mathbf{d}} \tag{2A.12}$$

Electric dipole moment defined

so that the torque on the dipole is given by:

$$\boxed{\mathbf{T} = \mathbf{p} \times \mathbf{E}} \tag{2A.13}$$

The torque experienced by an electric dipole in an electric field

The **d** (and therefore the **p**) points from negative to positive charge.

A dipole experiences no net force in a uniform field. It only experiences rotation.

## Effect of Boundaries

Changes in material affect an electric field

When an electric field goes from one material to another, what happens? Firstly consider what happens to a perpendicular **D** field:

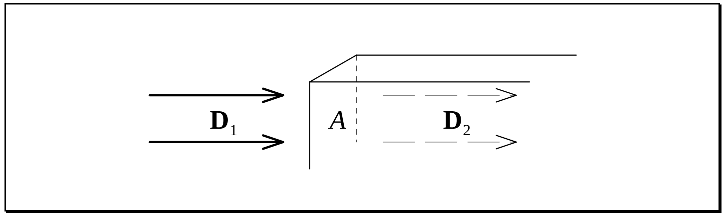



**Figure 2A.9**

There are no *free* sources of flux, since the material has no net charge. The flux outside and inside the material should be the same.

We can then say:

A perpendicular **D** field is not affected by a boundary

$$\begin{aligned}\psi_1 &= \psi_2\\ D_1 A &= D_2 A\\ D_1 &= D_2\end{aligned} \quad (2A.14)$$

Now consider what happens to a tangential **E** field:

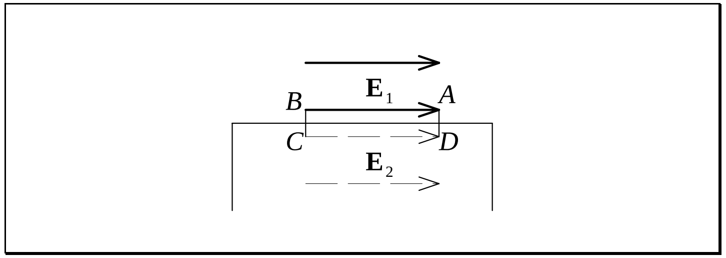



**Figure 2A.10**

The **E** field is conservative, so that:

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = 0 \quad (2A.15)$$

Performing this integral around the path *ABCD* gives (the distances *BC* and *AD* are assumed to be very small):

$$\begin{aligned}&\int_A^B \mathbf{E}_1\cdot d\mathbf{l} + \int_C^D \mathbf{E}_2\cdot d\mathbf{l} =\\ &\int_A^B E_1 dl - \int_C^D E_2 dl = 0\end{aligned} \quad (2A.16)$$

The integrals are easy to perform since the electric field is constant over the small distance *AB*. The result is:

A tangential **E** field is not affected by a boundary

$$\begin{aligned}E_1 l - E_2 l &= 0\\ E_1 &= E_2\end{aligned} \quad (2A.17)$$





In general, a boundary doesn't affect the perpendicular component of **D** and the tangential component of **E**.

We can now consider what happens when an electric field enters another material at an angle:

The effect of a boundary when the field is at an angle

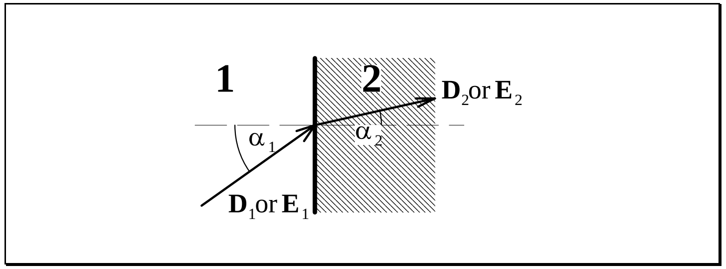



**Figure 2A.11**

We know from the previous results that:

$$E_1 \sin\alpha_1 = E_2 \sin\alpha_2 \quad (2A.18a)$$

$$D_1 \cos\alpha_1 = D_2 \cos\alpha_2 \quad (2A.18b)$$

$$D = \varepsilon_r \varepsilon_0 E$$

which leads to:

The direction and magnitude of the field changes

$$\frac{\varepsilon_{r1}}{\varepsilon_{r2}} = \frac{\tan\alpha_1}{\tan\alpha_2} \quad (2A.19)$$

It is interesting to note that we can apply the same reasoning to current in a conductor. In that case we can simply replace $D$ and $\varepsilon$ by $J$ and $\sigma$.

## Polarisation

A dielectric changes the permittivity

Why does the permittivity vary from that of free space for dielectrics? Consider first a parallel-plated capacitor with an air dielectric:

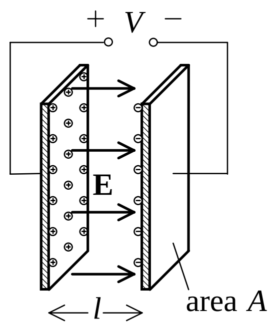



**Figure 2A.12**

The applied potential $V$ forces positive and negative charges to flow to the metal plates till $-\int_l \mathbf{E}\cdot d\mathbf{l} = V$ . As the field is uniform (the lines of force are parallel to each other), inside the capacitor **E** and **D** are constant. (This ignores fringing – a valid assumption for a small distance $l$).

Therefore, by performing the above integral, we get:

The fields in air for a parallel plated capacitor

$$\begin{aligned} E_0 &= V/l \\ D_0 &= \varepsilon_0 E_0 \end{aligned} \qquad (2A.20)$$

This is valid only in between the capacitor plates.

We define the capacitance of the structure as:

Capacitance defined

$$C = \frac{\text{electric flux}}{\text{electric potential}} = \frac{\Psi}{V} = \frac{D_0 A}{E_0 l} = \frac{\varepsilon_0 A}{l} \qquad (2A.21)$$





The effect of a dielectric in a parallel plated capacitor

What happens when we insert a slab of dielectric material between the metal plates? The molecules in the dielectric become polarised, and the electric dipoles will align in the direction of **E**:

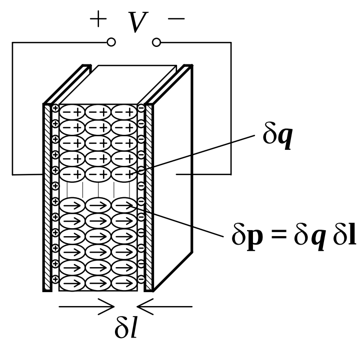



**Figure 2A.13**

A polarised dielectric can be modelled using the surface charges

In the material the charges of the dipoles will cancel, leaving (in effect) only the surface charges. The polarised slab is therefore equivalent to positive and negative surface charges which give rise to an internal field $\mathbf{E}_i$ . The direction of this induced **E** field opposes the applied field.

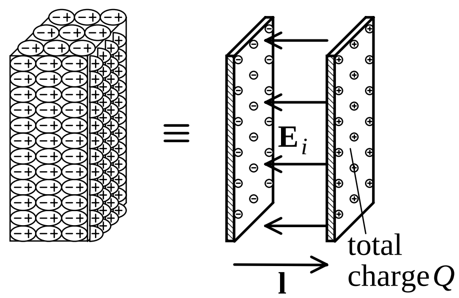



**Figure 2A.14**

The surface charges look like one big dipole:

$$\mathbf{p} = Q\mathbf{l} \tag{2A.22}$$

where $Q$ is the total charge on one surface, $l$ is the distance between them.

We define polarisation as the dipole moment per unit volume:

Polarisation defined

$$\boxed{\mathbf{P} = \frac{\mathbf{p}}{V}} \tag{2A.23}$$

and in this case we have:

and applied to a dielectric in a parallel plate capacitor

$$\mathbf{P} = \frac{\mathbf{p}}{V} = \frac{Q\mathbf{l}}{Al} = \frac{Q}{A}\hat{\mathbf{l}} \tag{2A.24}$$

The induced electric flux density between the two charge layers is:

Polarisation gives rise to an induced field

$$\mathbf{D}_i = -\frac{Q}{A}\hat{\mathbf{l}} = -\mathbf{P} \tag{2A.25}$$

and also induces free charges onto the capacitor plates

The applied voltage is constant which means the magnitude of the electric field inside the dielectric is fixed by the voltage source (see Eq. (2A.20)). To keep the electric field the same, more charges must be deposited onto the metal plates by the voltage, to compensate for the polarisation effect. The resultant magnitude of the flux density inside the slab is given by:

The relationship between **D** and **E** for a dielectric

$$\begin{aligned} D_d &= D_0 - D_i \\ &= \varepsilon_0 E_0 + P \\ &= \varepsilon_d E_0 \end{aligned} \tag{2A.26}$$





We define the dielectric permittivity:

$$\varepsilon_d = \varepsilon_0 + \frac{P}{E_0} \tag{2A.27}$$

Permittivity defined in terms of polarisation

and relative permittivity:

$$\boxed{\varepsilon_r = \frac{\varepsilon_d}{\varepsilon_0}} \tag{2A.28}$$

Relative permittivity defined (again)

If the applied **E** reverses periodically, **P** tries to keep up with it – the dipoles "flip". At very high frequencies, $\varepsilon$ decreases as the dipoles are unable to follow the applied **E**.

Frequency affects polarisation

## Ferroelectrics

Ferroelectrics have domains of permanent polarisation

Some materials exhibit permanently polarised regions (called domains):

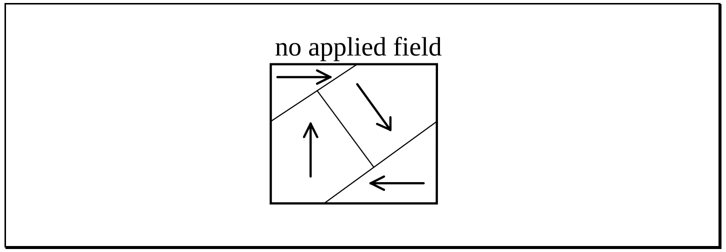



**Figure 2A.15**

An external field causes the domains to grow

An externally applied **E** field causes the growth of domains in its direction:

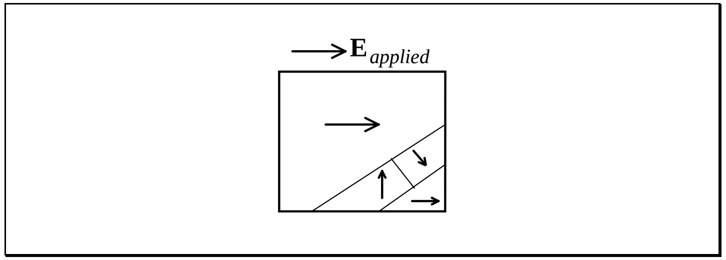



**Figure 2A.16**

A first look at hysteresis

The process is not reversible (hysteresis):

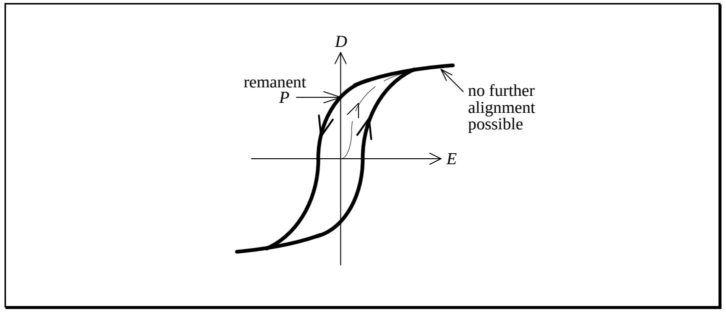



**Figure 2A.17**

When the applied **E** field is reduced to zero, a remanent **P** remains (we then have an electret – similar to a magnet).





## Breakdown at Sharp Points

Sharp points accumulate free charge

Charges tend to accumulate at sharp points on a conducting body. The charge tries to spread itself out as much as possible over the surface of the conductor. A sharp point is a long way from most of the surface.

Consider two conducting spheres connected by a perfect conductor (carrying no surface charge) and *assume that all the surface charge on the metal spheres is concentrated at their centre*. Also, the spheres are sufficiently far apart so that *proximity effects are negligible*.

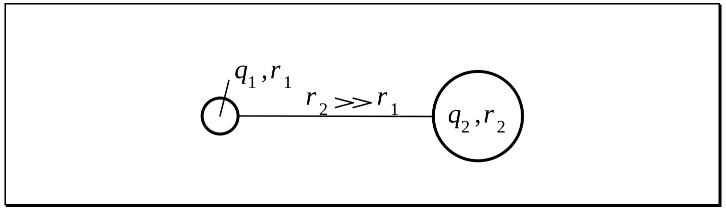



**Figure 2A.18**

The electric potential at the surface of sphere 1 is:

$$V_1 = -\int_{\infty}^{r_1} \mathbf{E}_1 \cdot d\mathbf{l} = -\int_{\infty}^{r_1} \frac{q_1}{4\pi\varepsilon_0 r^2} dr = \frac{q_1}{4\pi\varepsilon_0 r_1} \tag{2A.29}$$

Similarly:

$$V_2 = \frac{q_2}{4\pi\varepsilon_0 r_2} \tag{2A.30}$$

As the surface of a conductor is an equipotential surface (*Why?*), all surfaces are at the same potential and $V_1 = V_2$. Then:

$$\frac{q_1}{r_1} = \frac{q_2}{r_2} \quad \text{or} \quad \frac{4\pi r_1^2 D_1}{r_1} = \frac{4\pi r_2^2 D_2}{r_2} \tag{2A.31}$$

and since $D = \varepsilon_0 E$, this becomes:

$$r_1 E_1 = r_2 E_2 \quad \text{or} \quad \frac{E_1}{E_2} = \frac{r_2}{r_1} \tag{2A.32}$$

Therefore, at the surface of the spheres:

Sharp points have large fields around them

$$E_1 >> E_2 \quad \text{and} \quad D_1 >> D_2 \tag{2A.33}$$

This analysis would apply to shapes such as:

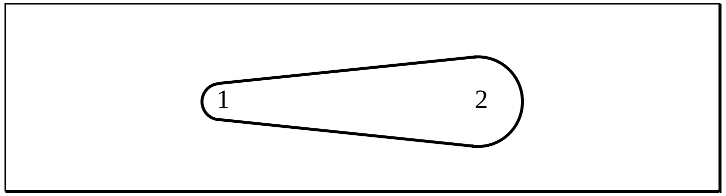



**Figure 2A.19**

The mechanism of insulator breakdown for air

In air (which is an insulator), molecules in the immediate vicinity of sharp points on the surface of a conductor will be under a high electric stress (a strong **E** field of high intensity as well as a high electric potential). Free charges (electrons and ions) are accelerated and acquire enough kinetic energy to displace bound electrons by collision with neutral air molecules. This results in large numbers of free charges. The surrounding air becomes conducting and the metal surface discharges (i.e. the *insulation breaks down*). This is the principle behind the lightning rod.

The insulation breakdown may be in the form of:

Low current breakdown in the form of a corona

(i) *Corona discharge* – low current, high potential. Hissing noise and glow results (energy emitted due to collisions).

High current breakdown in the form of an arc

(ii) *Arc discharge* – complete breakdown in the air path between two charged objects kept at different potentials. A large current results. The potential is very low.





**Example – Parallel plate capacitor**

It is useful to examine the various field quantities as a function of position inside a capacitor. Firstly consider what happens when a dielectric is introduced into a parallel plate capacitor when the voltage across the plates is kept constant:

Fields in a parallel plate capacitor when $V$ is constant

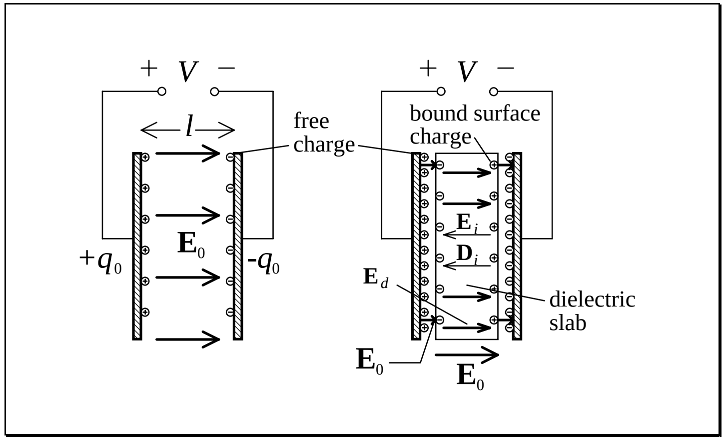



**Figure 2A.20**

With the dielectric inserted the system is equivalent to:

Equivalent circuit considering air gaps

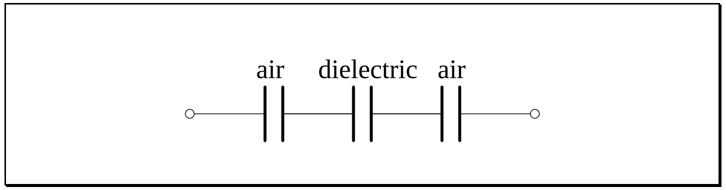



**Figure 2A.21**

The applied voltage $V$ is held constant. Inside the dielectric, the “bound charge” rearranges itself for a short time as polarisation **P** takes place.

Inside the dielectric, the total electric field magnitude is:

$$E_d = E_0 - E_i \qquad \left(P = -\varepsilon_0 E_i\right) \tag{2A.34}$$

If the dielectric is isotropic (same in all directions), then:

$$D_d = D_0^{new} = \varepsilon_0 E_0 + P = \varepsilon_d E_0 \tag{2A.35}$$

Permittivity accounts for all polarisation effects

The permittivity $\varepsilon_d$, which can be determined experimentally, accounts for all polarisation effects.

As $V$ is constant, when the dielectric is introduced additional free charge is induced on the capacitor plates to balance the polarisation equivalent surface charge.





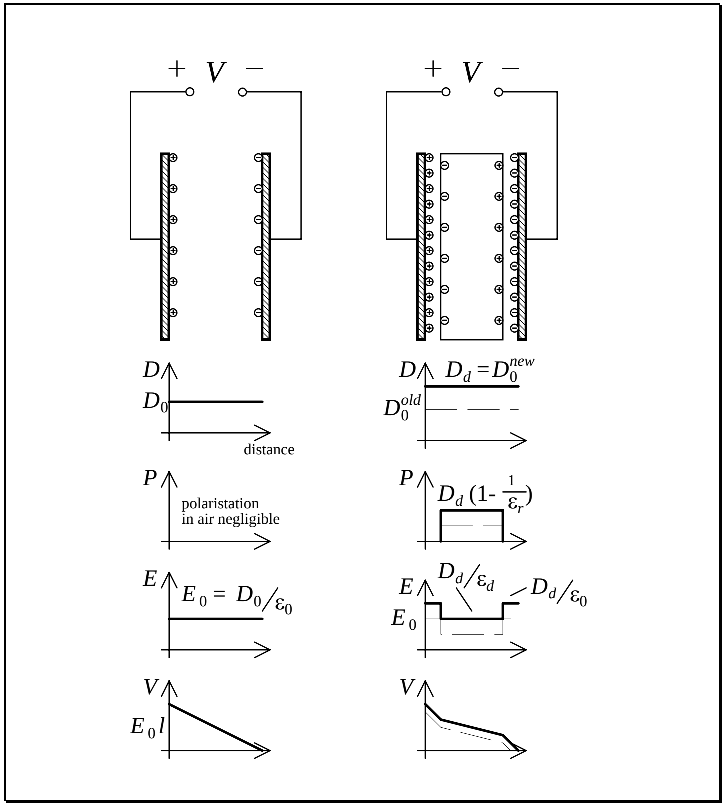



**Figure 2A.22**

The other possible way of introducing the dielectric is to keep $Q$ constant instead of $V$ (--- in Figure 2A.22). After charging the capacitor, $V$ is removed so there is no way $Q$ can change. When the dielectric is introduced, $Q = CV$ applies and the potential decreases (also $\mathbf{E}_d < \mathbf{E}_0$).

If $Q$ is constant, the fields are different

**Example – Air cavities in dielectrics and partial discharges**

Field caused by an air cavity in a dielectric

Consider a spherical air cavity inside a dielectric:

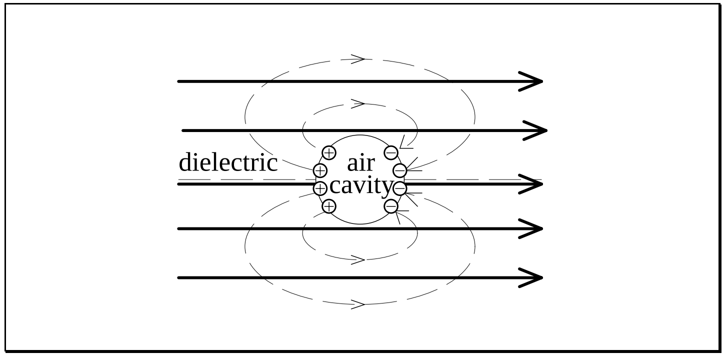



**Figure 2A.23**

The polarisation in air is negligible. The net charge at the boundary (caused by the dielectric's dipoles) acts as a spherical dipole with a field opposing the main field ($\varepsilon_d > \varepsilon_0$) in the dielectric. Therefore:

$$D_d > D_0, \quad \varepsilon_d E_d > \varepsilon_0 E_0 \tag{2A.36}$$

Also, charge concentration on the surface of the air cavity is large. Therefore:

The **E** field is larger in the air cavity than in the dielectric

$$E_0 > E_d \tag{2A.37}$$

and breakdown in the air may occur. (N.B. For a plane boundary $\varepsilon_d E_d = \varepsilon_0 E_0$, and since $\varepsilon_d >> \varepsilon_0$, then $E_0 >> E_d$).





The resultant field looks like:

The resultant **E** field of a dielectric caused by an air cavity

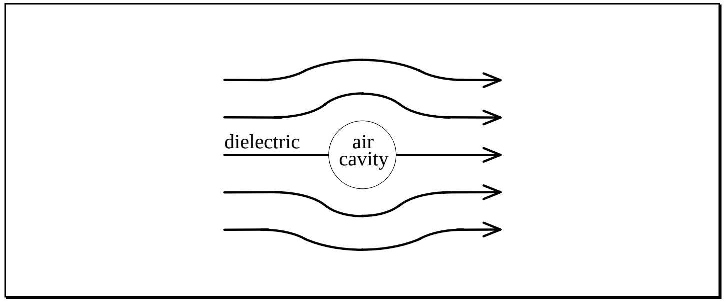



**Figure 2A.24**

*Draw the* **E** *and* **D** *fields in the air cavity.*

The electrical equivalent circuit of an air cavity in a dielectric is shown below:

Electrical equivalent circuit of the system

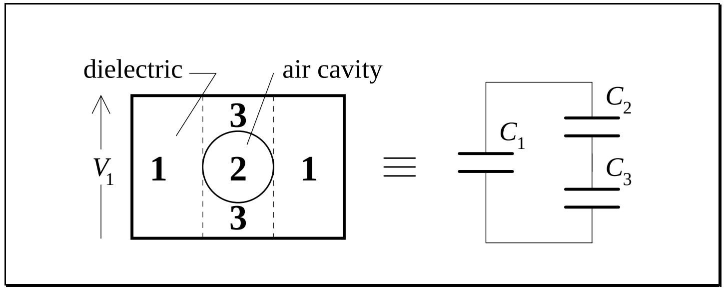



**Figure 2A.25**

The electrical equivalent circuit gives:

$$V_1 = V_2 \frac{C_2 + C_3}{C_3} \tag{2A.38}$$

Internal voltages can be estimated by the external voltages

and we can estimate the internal discharge voltages from changes in $V_1$.

A dielectric in air

If the dielectric sphere has a larger permittivity than the surrounding medium, then the fields are different:

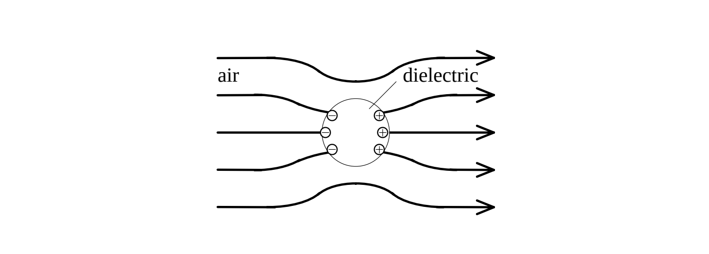



**Figure 2A.26**

*Draw the fields in the dielectric.*

Here, we have:

The **E** field in the dielectric is smaller than in the air

$$D_0 < D_d, \quad E_0 > E_d \tag{2A.39}$$

How to measure **E** and **D** in a dielectric

To measure **D** and **E** inside a dielectric, we can use a probe that measures $\delta V = E\delta l$. To measure **D** we use a disc shaped cavity, to measure **E** we use a needle shaped cavity:

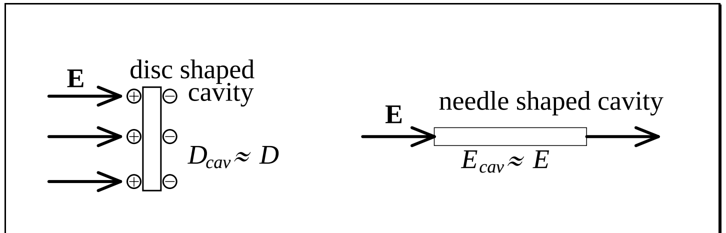



**Figure 2A.27**





**Example – Variable permittivity capacitor**

A capacitor with varying permittivity is modelled

A capacitor has a dielectric with an $\varepsilon_r$ that varies linearly with distance. Determine the capacitance per unit area.

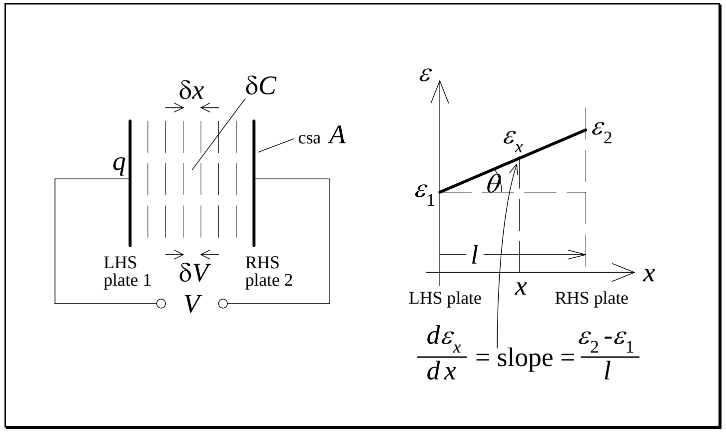



**Figure 2A.28**

We have:

as lots of capacitors in series

$$V = \sum \delta V = q \sum \frac{1}{\delta C} \quad \text{where} \quad \delta C = \frac{\varepsilon_x A}{\delta x} \tag{2A.40}$$

Therefore:

$$\begin{aligned} V &= q\int_1^2 \frac{dx}{\varepsilon_x A} = \frac{q}{A}\int_1^2 \frac{l}{\varepsilon_2 - \varepsilon_1}\frac{d\varepsilon_x}{\varepsilon_x} \\ &= \frac{ql}{A(\varepsilon_2 - \varepsilon_1)}\left[\ln \varepsilon_x\right]_{\varepsilon_1}^{\varepsilon_2} = \frac{ql}{A(\varepsilon_2 - \varepsilon_1)} \ln\frac{\varepsilon_2}{\varepsilon_1} \end{aligned} \tag{2A.41}$$

Finally, we get the capacitance per unit area:

$$\frac{C}{A} = \frac{\varepsilon_2 - \varepsilon_1}{l \ln\left(\varepsilon_2 / \varepsilon_1\right)} \ \text{Fm}^{-2} \tag{2A.42}$$

**Example – Electrostatic generator (Van de Graaff generator)**

Consider a Van de Graaff generator, with a dome radius $r = \frac{1}{3}$ m, and an equivalent current source $I = 1\,\mu\text{A}$:

The Van de Graaff generator produces a large static voltage

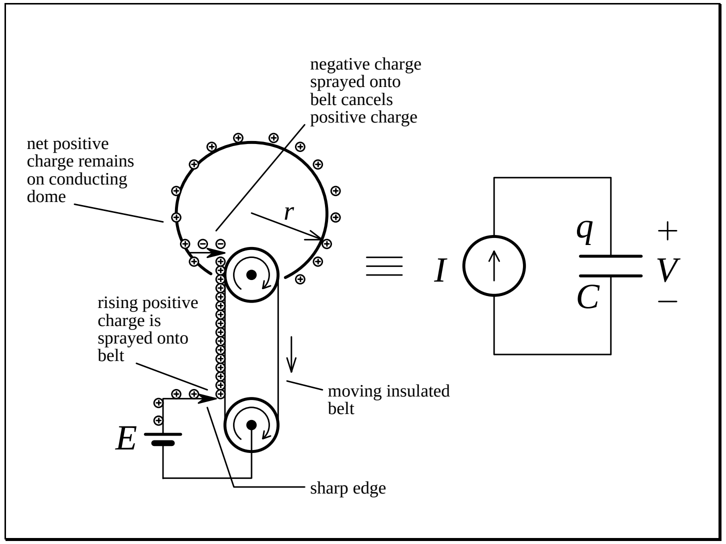



**Figure 2A.29**

If we approximate the dome by a sphere, then:

$$V = -\int_{\infty}^{r} \mathbf{E} \cdot d\mathbf{r} = -\int_{\infty}^{r} \frac{q}{4\pi\varepsilon_0 r^2} dr = \frac{q}{4\pi\varepsilon_0 r} \qquad \text{(2A.43)}$$

If $E_B$ = electric field breakdown strength ($\approx 3\times10^6$ Vm$^{-1}$ for air, plane-plane) then:

(i) maximum dome voltage is: $V_{max} = E_B r = 3\times10^6 \times \frac{1}{3} = 10^6$ V

(ii) dome capacitance is: $C = \frac{q}{V} = 4\pi\varepsilon_0 r = 4\pi \times 8.85\times10^{-12} \times \frac{1}{3} \approx 37$ pF





(iii) time to reach breakdown is:

$$i = C\frac{dv}{dt}$$

$\therefore$ change in voltage is $\delta V = \frac{I}{C}\delta t$

$$t_B = \frac{V_{max} C}{I} = \frac{10^6 \times 37 \times 10^{-12}}{1\times10^{-6}} = 37 \text{ s}$$

but eventually reaches breakdown

**Example – Lord Kelvin's Water Dynamo**

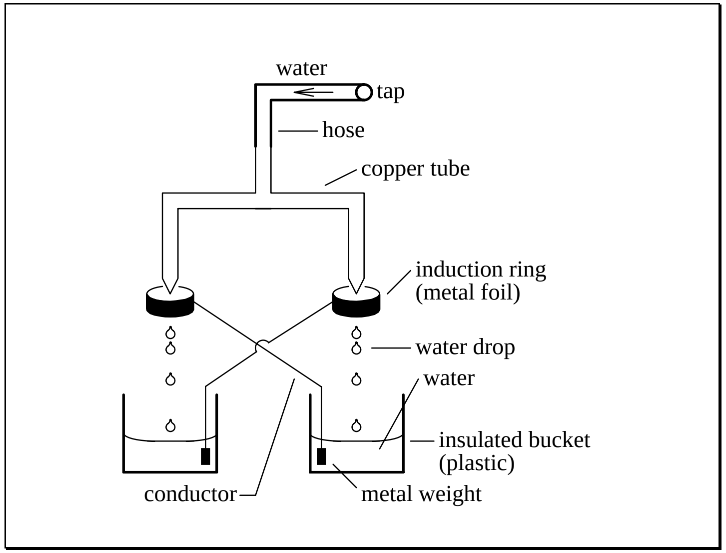



**Figure 2A.30**

## Summary

- The Bohr model of the atom leads to the concept of the energy barrier, and of free and bound charges. This concept then leads to an understanding of conductors and insulators.

- Resistivity is an intrinsic property of a conductor, and it varies with temperature.

- An electric dipole consists of equal positive and negative charges held a distance apart. An electric dipole experiences no net force in a uniform field. It only experiences rotation.

- When an electric field goes from one material to another, a perpendicular **D** field is not affected, nor is the tangential **E** field.

- A dielectric placed in an electric field produces a polarisation, which acts to increase the relative permittivity – for a capacitor, the capacitance will increase due to the polarisation.

- Sharp points of metallic objects accumulate free charge and have large electric fields around them – which can lead to *insulation breakdown*.

- Air cavities in dielectrics can produce *partial discharge*.

## References

Plonus, Martin A.: *Applied Electromagnetics*, McGraw Hill Kogakusha, Ltd., Singapore, 1978.

Shadowitz, Albert: *The Electromagnetic Field*, Dover Publications, Inc., New York, 1975.

# Lecture 2B – Magnetism

*Magnetic dipole moment. Magnetisation. Diamagnetism. Paramagnetism. Ferromagnetism. The* B-H *characteristic (hysteresis). The normal magnetization characteristic.*

## Magnetic Dipole Moment

The magnetic field produced by a loop of wire (obtained using the Law of Biot-Savart) looks similar to that of a magnet. Therefore, a current loop can be considered to be like a small permanent magnet, and it will have a magnetic dipole moment (similar to the electric dipole moment).

A current loop has a magnetic field like a permanent magnet

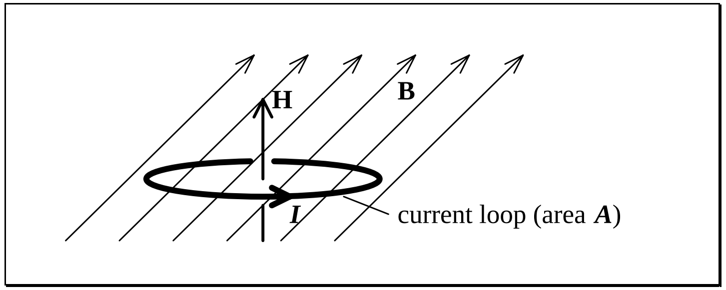



**Figure 2B.1**

In an external field **B,** the effect of the Lorentz Force on each element of the current loop is to produce a torque:

$$\mathbf{T} = I\mathbf{A} \times \mathbf{B} \tag{2B.1}$$

The torque on a current loop immersed in a **B** field

To make this similar to the torque experienced by an electric dipole, we define the magnetic dipole moment to be:

$$\mathbf{m} = I\mathbf{A} \tag{2B.2}$$

Magnetic dipole moment defined

The torque experienced by a current loop due to an external field can then be expressed as:

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \tag{2B.3}$$

The torque experienced by a magnetic dipole in a magnetic field

This torque will tend to align a magnetic dipole in the direction of an applied field.

Electron orbital motion is a current

An atom with an orbiting electron can be modelled as a current loop.

A permanent magnet has many dipoles in the same direction

A permanent magnet is made of many molecular magnetic dipole moments that align in the same direction:

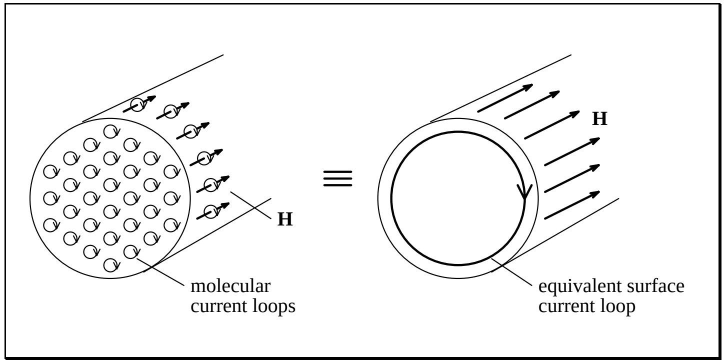



**Figure 2B.2**

This is why the field of a solenoid looks like that of a magnet.

## Magnetisation

The tiny magnets created by circulating atomic currents are the sources of the **B** field of permanent magnets and magnetisable materials. Experiment shows that certain materials (called magnetic materials), when placed in a magnetic field, react upon it and modify it. This phenomenon is called magnetisation. The magnetization is defined as the average dipole moment per unit volume:

Magnetisation defined

$$\mathbf{M} = \frac{\mathbf{m}}{V} \tag{2B.4}$$

Remember the concept of polarization. The magnetization provides a link between the microscopic (**m**) and the measurable (**M**).





Magnetisation contributes to the external field

A magnetic material that is placed in a magnetic field will become magnetized. The material then contributes to the external field. A measure of this induced effect (like polarization) is the magnetization. In ferromagnetic materials, the induced **M** remains after the external field is withdrawn.

This explains why a rod of steel that is inserted into a solenoid increases the field.

The magnetic field **B** is modified by the induced **M**:

The relationship between **B** and **H** for a magnetisable material

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$$
$$= \mu\mathbf{H}$$

Permeability defined in terms of magnetisation

$$\mu = \mu_0\left(1 + \frac{\mathbf{M}}{\mathbf{H}}\right) = \mu_0\mu_r \tag{2B.5}$$

Magnetic materials are classified into three groups:

(i) diamagnetic ( $\mu_r \approx 0.999$ ). eg. molecular hydrogen, water, copper, glass.

(ii) paramagnetic ( $\mu_r \approx 1.001$ ). eg. molecular oxygen, aluminium.

(iii) ferromagnetic ( $\mu_r \geq 100$ ). eg. iron, nickel, cobalt.

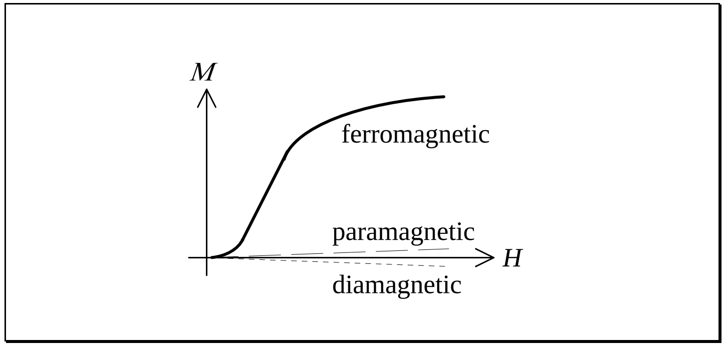



**Figure 2B.3**

Ferromagnetism disappears at high temperatures

Above a certain temperature $T_C$ (called the Curie temperature), ferromagnetism disappears and ferromagnetic materials become paramagnetic.

There are two possible causes of magnetism:

(i) electron orbital motion around the nucleus.

(ii) electron spin (about own axis).

## Diamagnetism

Diamagnetism is an induced effect caused by orbiting electrons

Diamagnetism is essentially a quantum mechanical phenomenon. To do a "classical" analysis that agrees with observed results, we have to assume that electrons are paired in orbits and move in opposite directions at the same speed. Without an applied field, there is no *net* magnetic moment.

Consider one orbiting electron:

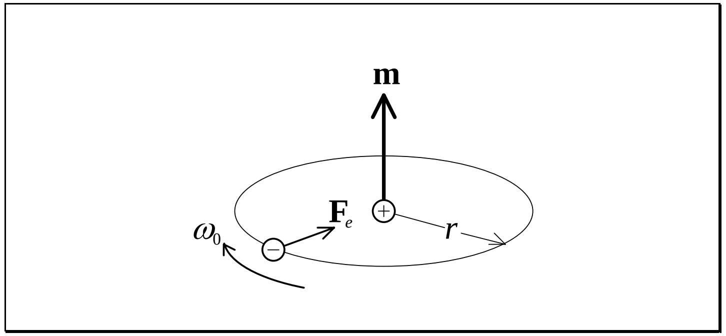



**Figure 2B.4**

The electron is in equilibrium in its orbit. An electric centripetal force holds the electron to its atom:

$$F_e = m_e a = m_e \omega_0^2 r \tag{2B.6}$$

An electron in an **H** field experiences a Lorentz Force

Application of a magnetic field **H** exerts an additional magnetic force on the electron (a Lorentz force). The radius of the electron orbit does not change, since we are using the Bohr model of the atom. The direction of the Lorentz





force depends on the direction of the magnetic field. Assume an **H** field direction that slows down the electron:

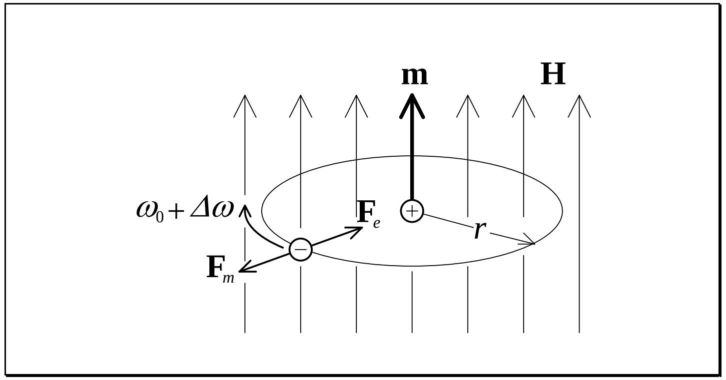



**Figure 2B.5**

Newton's second law gives, for the new angular velocity:

$$\begin{aligned} F_e - F_m &= m_e a \\ m_e \omega_0^2 r - e\omega r \mu_0 H &= m_e \omega^2 r \\ -e\omega\mu_0 H &= m_e(\omega - \omega_0)(\omega + \omega_0) \end{aligned} \tag{2B.7}$$

Since the change in speed will be small, then:

$$\begin{aligned} \omega - \omega_0 &= \Delta\omega \\ \omega + \omega_0 &\approx 2\omega \end{aligned}$$

An electron in an **H** field changes speed in proportion to the field strength

$$\Delta\omega \approx -\frac{e\mu_0}{2m_e} H \tag{2B.8}$$

Diamagnetism reduces the **B** field

The decrease in electron speed is proportional to the applied field. The electron orbiting in the opposite direction would speed up. The resultant effect is to reduce the field in the material.

Ferromagnetic materials have domains

## Paramagnetism

Electrons not only have orbital motion, but spin motion as well.

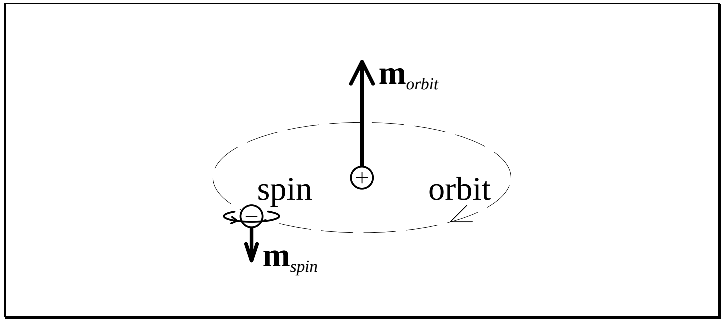



**Figure 2B.6**

Paramagnetism is an effect caused by spinning electrons

Each spinning electron produces a spin magnetic moment. Due to thermal vibrations, the axes of the spins are randomly distributed over all possible orientations. A piece of paramagnetic material has no net external magnetization.

Paramagnetism is an alignment of the spin magnetic moments

An applied **H** field will tend to align theses magnetic moments in its direction. The alignment is opposed by thermal agitation which for paramagnetic and diamagnetic materials is much stronger. The result is a very slight increase in the magnetic field in the material.

## Ferromagnetism

Ferromagnetism is an effect caused by unpaired spinning electrons

Inner shells (close to the nucleus) of a ferromagnetic atom have unpaired electrons, which are shielded from the influence of other atoms. Each molecule therefore exhibits a strong resultant spin magnetic moment. The strong field of the molecular dipoles causes them to align over small volumes called domains. (A domain has a typical dimension between $10^{-3}$ and $10^{-6}$ m, and contains about $10^{16}$ atoms. They were discovered by Weiss in 1906).

Normally the domains are oriented at random and are not noticeable externally. When an external **H** field is applied, the dipoles try to align with **H** and domains with **M** in the direction of **H** grow at the expense of the others.





Ferromagnetic materials give rise to a large increase in the resultant field

“Saturation” is reached when no further dipole alignment is possible. A strong **B** field results. On removal of the applied field, some magnetization is retained (the domains do not return to their original state).

## The *B*-*H* Characteristic (Hysteresis)

A piece of ferromagnetic material without any applied fields has the following microscopic structure:

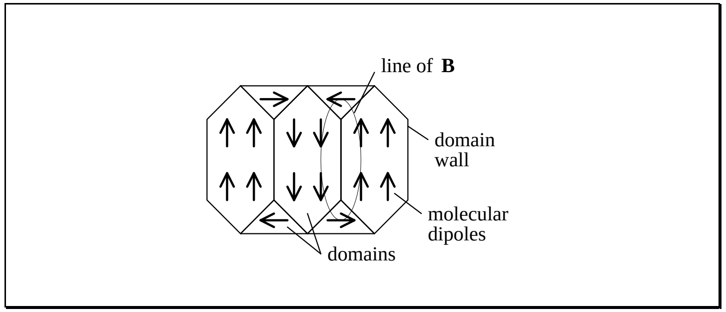



**Figure 2B.7**

The domain structure of ferromagnetic material

There are large internal fields that cause the molecular dipoles to align in regions called domains. Adjacent domains are oriented so that the magnetic field lines form closed loops *easily* (using the minimum of energy).

If we apply a large external **H** field to the material then three things happen:

The dipoles firstly align in a direction of "easy" magnetisation

(i) magnetic dipoles tend to align with the applied field in directions of "easy" magnetisation (those directions that line up with the crystal structure of the material). Removal of the field causes the dipoles to turn back to their original state – the process is reversible:

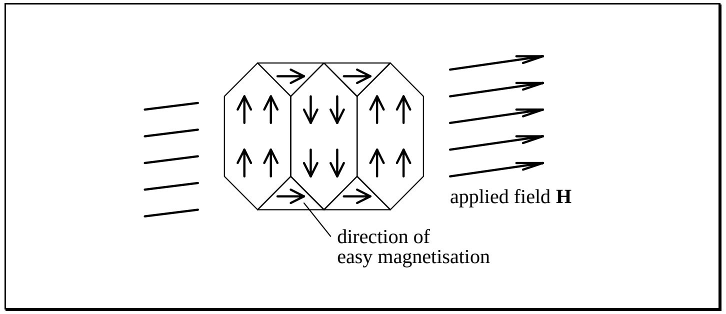



**Figure 2B.8**

Then the domains in the direction of **H** grow at the expense of other domains

(ii) domains in the general direction of the applied field grow at the expense of others. This involves movement of the domain walls – it takes energy and is irreversible. Eventually, there is just one domain:

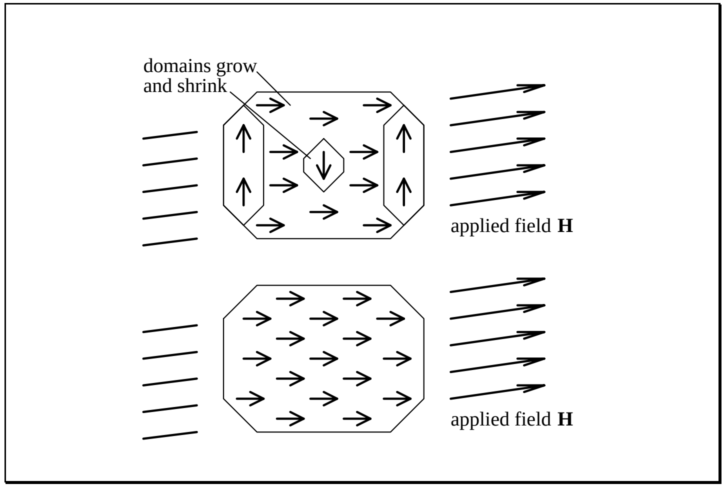



**Figure 2B.9**





(iii) the magnetic dipoles align with the applied field (called "hard" magnetisation because generally the field does not line up with the crystal structure of the material) until all dipoles are aligned – saturation has been achieved.

Finally, the dipoles turn in the direction of "hard" magnetisation

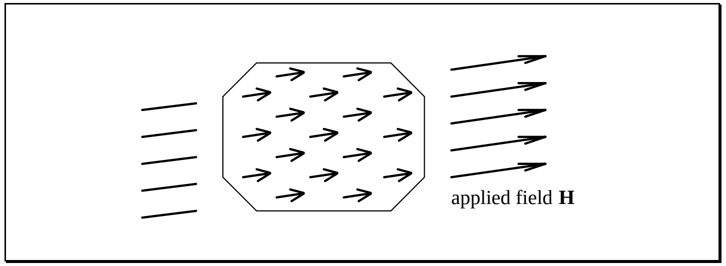



**Figure 2B.10**

This magnetisation process is shown macroscopically by a *B*-*H* characteristic:

Features of the *B*-*H* characteristic

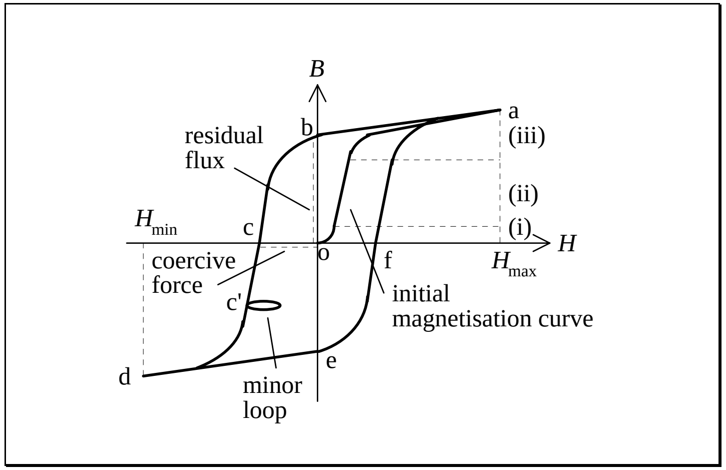



**Figure 2B.11**

To observe the way a *B-H* characteristic is traced out, we can use a toroidal specimen (*Why a toroid?*) and direct current:

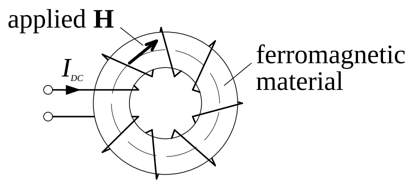



**Figure 2B.12**

The way in which a *B-H* characteristic is traced out

The steps to obtain the *B-H* characteristic are:

(i) **H** (or $I_{DC}$) is gradually increased. **B** in the material increases along *oa* until no further alignment is possible (saturation is reached at $H_{max}$).

(ii) **H** (or $I_{DC}$) is reduced. **B** decreases along *ab* (not *ao*). This property is known as hysteresis (Greek: short coming). No part of the magnetization curve is now reversible.

(iii) **H** is further reduced to $H_{min}$ and then increased again. **B** follows the path *bcdefa*. (N.B. the path terminates at *a* only if we apply $H_{max}$ again).

### Minor Loops

Minor loops defined

If we are at point *c'* (say) and **H** is increased (made more positive) and then decreased to the previous value, the minor loop shown in Figure 2B.11 is traced.





## The Normal Magnetization Characteristic

Different *B-H* loops are obtained for different values of $H_{max}$. Joining the tips of each hysteresis loop gives the "normal magnetization characteristic":

The normal magnetization characteristic defined

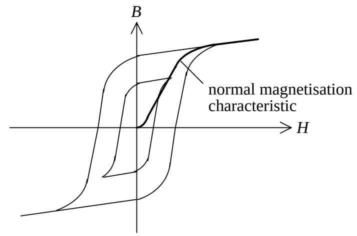



**Figure 2B.13**

The normal magnetization characteristic is used often. It is like an average characteristic, but it doesn't tell us about the shape of the hysteresis loop (and therefore the losses). It is well suited to analysis where AC excitation of the material is involved. (*Why?*)

and used instead of a hysteresis loop

## Summary

- A magnetic dipole is a current loop. A magnetic dipole experiences a torque when subjected to an external magnetic field.
- An atom with an orbiting electron can be modelled as a current loop, i.e. as a magnetic dipole.
- A magnetic dipole placed in a magnetic field experiences magnetisation, which acts to increase the relative permeability – for an inductor, the inductance will increase due to the magnetisation.
- Magnetic materials can be categorised into three groups: diamagnetic, paramagnetic and ferromagnetic.
- Diamagnetism is caused by orbiting electrons – it reduces the **B** field slightly.
- Paramagnetism is caused by spinning electrons – it increases the **B** field slightly.
- Ferromagnetism is cause by unpaired spinning electrons – it increases the **B** field significantly.
- A ferromagnetic material's magnetic properties can be described with a *B-H* characteristic, which exhibits hysteresis. The *B-H* characteristic is a result of the physical crystal structure which is divided into domains.

## References

Plonus, Martin A.: *Applied Electromagnetics*, McGraw Hill Kogakusha, Ltd., Singapore, 1978.

Shadowitz, Albert: *The Electromagnetic Field*, Dover Publications, Inc., New York, 1975.

Shamos, Morris H. (Ed.): *Great Experiments in Physics - Firsthand Accounts from Galileo to Einstein*, Dover Publication, Inc., New York, 1959.

# Lecture 3A – Semiconductors

*Semiconductor structure.* p-*type semiconductor.* n-*type semiconductor. The* p-n *junction. The* p-n *junction characteristic (diode* v-i *characteristic). Diode models. The Hall-effect device. Breakdown diodes. The photodiode. The light emitting diode (LED). The Schottky diode. The varactor diode.*

## Semiconductor Structure

The predominant semiconductor material is silicon. Silicon is one of the most abundant elements on Earth, and is always found in compound form in nature (sand is mainly $SiO_2$). It is purified by chemical means so that the concentration of troublesome impurities is about 1 in a billion. The valence of silicon is 4, like carbon. It is the valence electrons that participate in chemical bonding when the atoms form compounds. Silicon crystallizes in a diamond-like structure, because this minimizes the free energy – each atom has four neighbours, set in a tetrahedral structure.

Silicon is abundant and has a tetrahedral crystal structure

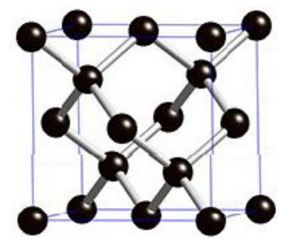


**Figure 3A.1**

Each of the 4 valence electrons is shared with a neighbour, which is called covalent bonding. It does not involve electric charge transfer between different locations in the lattice.

Pure crystalline silicon will possess the same structure as diamond, but it is nowhere near as hard a substance as diamond. There are two reasons for this. Firstly, the silicon-silicon bond is much weaker than the carbon-carbon bond (silicon is a bigger atom). Secondly, carbon is a significantly smaller-than-average atom, and there are vastly more bonds per unit of volume in a diamond than in any other substance.

To simplify things, we can describe the *Si* crystal in a two-dimensional form:

Silicon is held together with covalent bonds

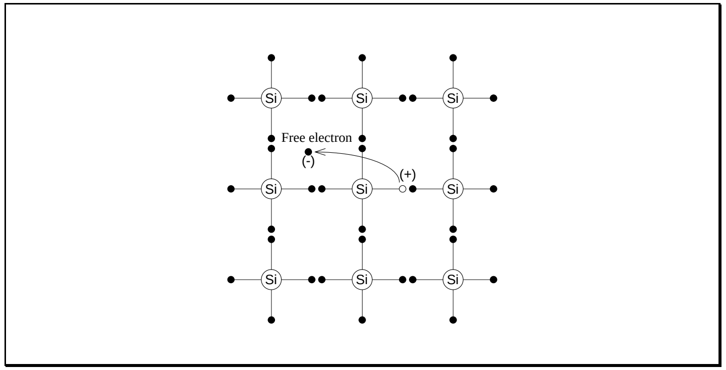



**Figure 3A.2**

At very low temperatures, pure silicon behaves as an insulator, since a shared electron is bound to its locality and there is no source of extra energy to free itself from its bonds and make itself available for conduction. The extra energy can be obtained from thermal vibrations of the crystal lattice atoms.

Thermal energy can break the covalent bonds, releasing an electron-hole pair

When a valence electron is freed, two charge carriers are created. The first is the electron itself. The second one, called a hole, is the charge located in the area vacated by the electron. That vicinity is left with a *net* positive charge (obviously caused by a silicon nucleus). Any one of the other valence electrons moving nearby can step into the vacated site. This shifts the *net* positive charge – the hole – to a new location. Both the free electron and the hole can therefore move around in the semiconductor crystal.

The conductivity of pure silicon is therefore proportional to the free carrier concentration, and is very small.





## *p*-type Semiconductor

Doping defined

To make devices like diodes and transistors, it is necessary to increase the electron and hole population. This is done by intentionally adding specific impurities in controlled amounts – a process known as doping.

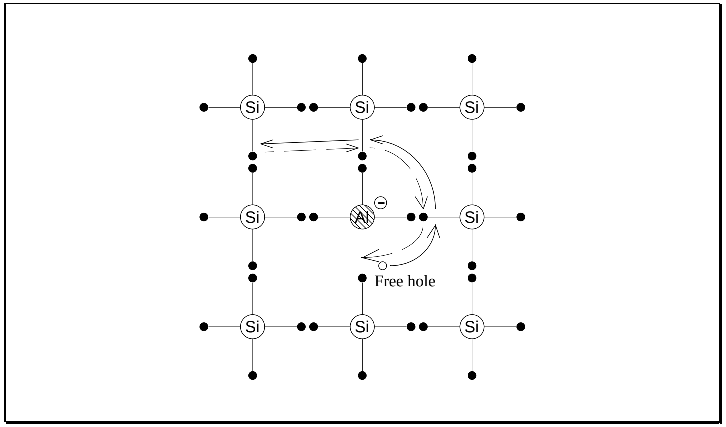



**Figure 3A.3**

If *Si* is doped with an element with only 3 valence electrons, then at the location of that impurity one of the covalent bonds is missing. The location is electrically neutral. One of the other valence electrons can cross over and complete the missing bond. When this happens a hole is created at the position vacated by that valence electron.

*p*-type semiconductor defined

The impurity atom, having accepted an additional electron, is called an acceptor and now has a net negative charge. A semiconductor doped with acceptors is rich in holes, i.e. positive charge carriers, and therefore called *p*-type.

In this case the holes are called the majority carriers, the electrons are called the minority carriers.

## *n*-type Semiconductor

If *Si* is doped with an element with 5 valence electrons, then four of the valence electrons will take part in the covalent bonding with the neighbouring *Si* atoms while the fifth one will be only weakly attached to the impurity atom location. The thermal energy of a semiconductor at room temperature is more than enough to free this electron, making it available for conduction.

*n*-type semiconductor defined

The impurity atom, having donated an additional electron, is called a donor. The semiconductor in this case is called *n*-type, because it is rich in negative charge carriers. The electrons are the majority carriers and the holes are the minority carriers for this type of semiconductor.

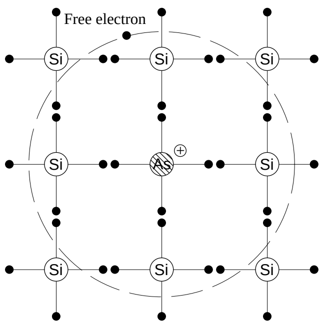



**Figure 3A.4**





## The *p-n* Junction

A *p-n*-junction is formed by doping a pure semiconductor

A *p-n* junction is a location in a semiconductor where the impurity type changes from *p* to *n*, while the lattice structure continues undisturbed. It is the most important region in any semiconductor device.

Initially assume a situation where the *p-n* junction is completely neutral. We can show each lattice site with a neighbouring free charge carrier:

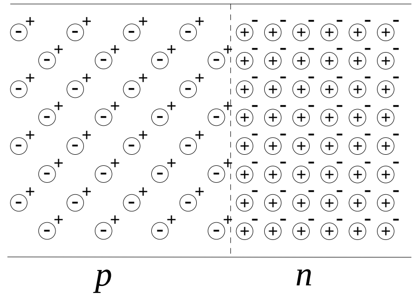



**Figure 3A.5**

A *p-n* junction is subject to diffusion of the majority carriers,

This is a very unnatural state of affairs. Imagine that we have a gas cylinder full of oxygen. We open the valve to release the oxygen into the room. *What happens?* The oxygen and the air in the room mix – a process known as diffusion. Nature wants things to spread out in an even fashion.

thus creating an electric field in the junction

This is what happens with the free, gas-like particles in the *p-n* junction. The holes and electrons will diffuse to try and cover the whole semiconductor in an even fashion. As soon as the holes and electrons move, they "uncover", or leave behind, a charge equal but opposite at that site in the crystal. This uncovered and fixed charge will create an electric field, according to Coulomb's law.

The process of recombination

Where do the holes and electrons go? Since they will try to diffuse across the *p-n* junction, they will find themselves in a region full of opposite charges. If an electron meets up with a hole, then *recombination* takes place. The electron "falls into" the hole so that the net charge is zero. They effectively disappear from our diagram.

The resulting electric field that exists at the *p-n* junction due to the "uncovered" charge forms a potential barrier. (*Refer back to Lecture 2A*). The field opposes further diffusion of electrons and holes. The field *does* cause drift currents of minority carriers. *Show how this happens*. At equilibrium, the two components of current exactly balance.

A depletion region is formed at the junction

A region exists, on both sides of the junction, in which there is a depletion of mobile carriers, since the field sweeps them away. This region is called the *depletion region*:

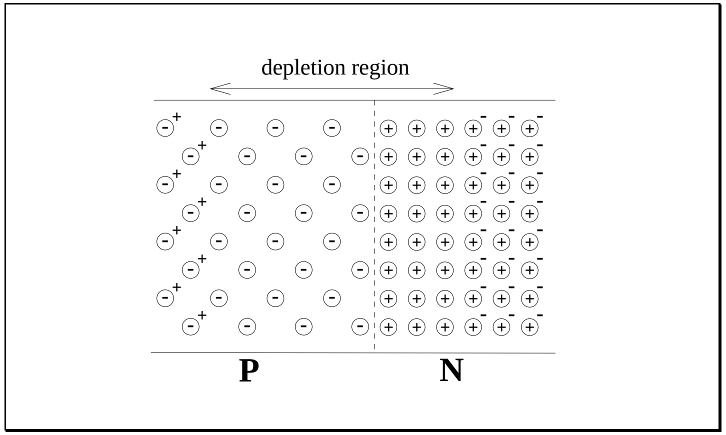



**Figure 3A.6**

A diode can be a *p-n* junction

A *p-n* junction forms what is called a diode. Its circuit symbol is:

The diode's circuit symbol

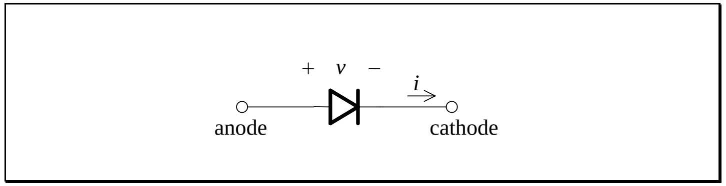



**Figure 3A.7**





Conditions inside the diode when open-circuited

Under open circuit conditions, the inside of the diode will have a potential barrier, as previously discussed:

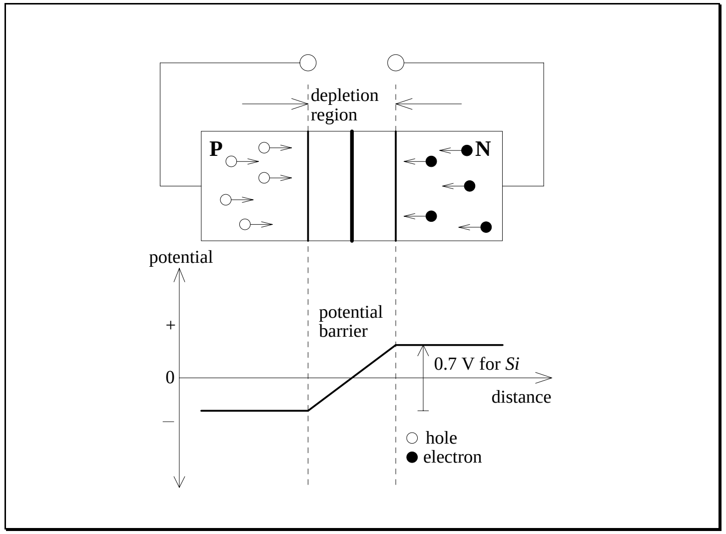



**Figure 3A.8**

Holes and electrons find it difficult to surmount the potential barrier - ball and hill analogy

To make an analogy, consider a slope at which we are rolling balls. Each ball will rise up the slope to a level where the gravitational potential energy equals its initial kinetic energy. If a ball has enough energy, it will reach the top of the slope (the "other" side). If a ball does not have enough energy, it will roll back down. You should be able to look at the diode and imagine electrons and holes trying to get over the barrier in this fashion.

The internal potential is not a voltage source

Is the internal potential (the barrier) available as a voltage source? *No*. For a diode, there are always metal contacts at the diode terminals which form a semiconductor-metal interface. These semiconductor-metal interfaces also create internal potential barriers, which cancel (or balance) the *p-n* junction's potential barrier. The result is an electrically neutral device, as expected.

**Reverse Bias**

Conditions inside the diode when reverse biased

When a voltage is applied as shown in Figure 3A.9, the diode is reverse biased. Majority carriers are taken out of the diode. The depletion region widens and the potential barrier increases. The diffusion current is very small. Leakage current in a diode biased in this fashion is due to drift of thermally generated minority carriers.

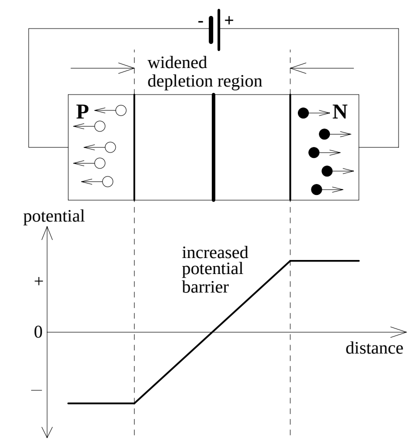



**Figure 3A.9**





**Forward Bias**

Majority carriers are supplied to the diode. The depletion layer gets smaller, and more charge carriers are able to overcome the reduced potential barrier. An increase in the diffusion current component results. The drift current remains the same.

Conditions inside the diode when forward biased

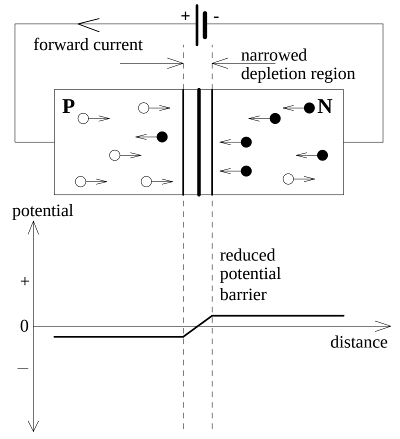



**Figure 3A.10**

**Junction Capacitance**

The depletion region of a diode effectively forms a capacitance (there are two conducting regions separated by a high permittivity region). *Show how the junction capacitance is dependent upon the depletion layer width.*

The diode's depletion region is a small capacitor

## The *p-n* Junction Characteristic (Diode *v-i* Characteristic)

The diode's terminal electrical characteristics can be obtained using the following circuit:

Obtaining a diode's terminal characteristics

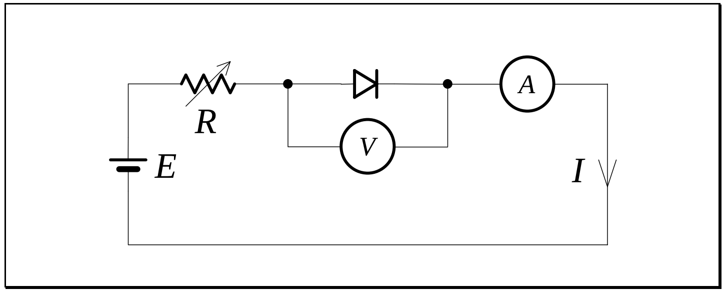



**Figure 3A.11**

With the battery as shown, we can vary $R$ and measure $V$ and $I$ to obtain the forward-bias characteristic. We could also use a curve tracer to obtain the characteristic. We can reverse the polarity of $E$ to obtain the reverse-bias characteristic. The total characteristic looks like:

A typical characteristic for a silicon diode

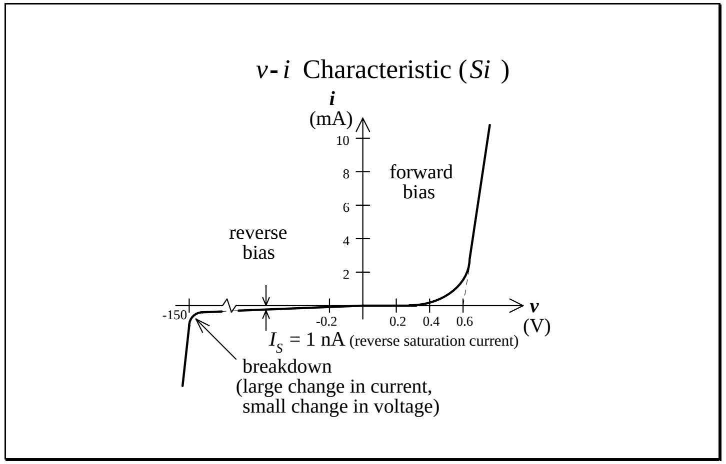



**Figure 3A.12**

The characteristic can be divided up into two main regions.





### Forward Bias

There is not much increase in current until the internal barrier voltage is overcome (approximately 0.6 V in silicon). Then large conduction results. The current is due to drift of the majority carriers and diffusion of the minority carriers. There is a "forward" capacitance associated with the diode. The thin depletion region gives rise to a "junction" capacitance, and the excess concentration of minority carriers on each side of the depletion region (caused by diffusion) gives rise to a "diffusion" capacitance. The diode's forward capacitance is then given by:

When forward biased, the diode conducts

$$C_{fd} = C_j + C_D \tag{3A.1}$$

where: $C_j$ = junction capacitance $(\approx \mu\text{F})$
$C_D$ = diffusion capacitance $(\approx \text{pF})$

### Reverse Bias

A small leakage current exists due to minority carriers. Before breakdown, the depletion region is very large so there is a small capacitance:

When reverse biased, the diode does not conduct

$$C_{rd} = C_j \tag{3A.2}$$

where: $C_j$ = junction capacitance $(\approx \text{pF})$

### Breakdown

If enough reverse bias is applied, the diode will "break down" and start conducting. It is not a destructive process unless the device cannot dissipate the heat produced in the breakdown process. Breakdown is actually exploited in certain types of diodes (e.g. the Zener diode) because of the near vertical characteristic in this region.

Breakdown occurs eventually for a large enough reverse bias

## Diode Models

*Why we model the diode*

The curve describing the diode's terminal characteristics is non-linear. How can we use this curve to do circuit analysis? We only know how to analyze linear circuits. There is therefore a need for a linear circuit model of the diode.

*The concept of modelling*

When we model something, we transform it into something else – usually something simpler – which is more amenable to analysis and design using mathematical equations. Modelling mostly involves assumptions and simplifications, and the only requirement of a model is for it to "work" reasonably well. By "work" we mean that it agrees with experimental results to some degree of accuracy.

Models are sometimes only valid under certain operating conditions, as we shall see when modelling the diode.

### The Ideal Diode Model

As a first approximation, we can model the diode as an ideal switch:

*The diode as an ideal (controlled) switch*

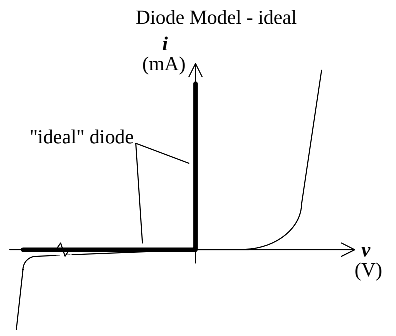



**Figure 3A.13**

The characteristic in this case is approximated by two straight lines – the vertical representing the "on" state of the diode, and the horizontal





representing the "off" state. To determine which of these states the diode is in, we have to determine the conditions imposed upon the diode by an external circuit. This model of the diode is used sometimes where a quick "feel" for a diode circuit is needed. The above model can be represented symbolically as:

*The ideal diode model*

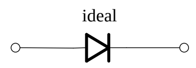



**Figure 3A.14**

**Example**

(i) Find the current, $I$, in the circuit shown below, using the ideal diode model.

(ii) If the battery is reversed, what does the current become?

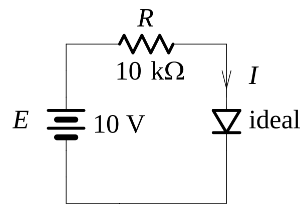
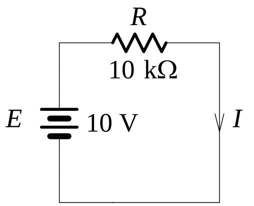



**Figure 3A.15**

(i) Firstly, we must determine whether the diode is forward biased or reverse biased. In this circuit, the positive side of the battery is connected (via the resistor) to the anode. Therefore, the anode is positive with respect to the cathode, and the diode is *forward biased*. In order to use the ideal diode model, the diode is simply replaced by the ideal diode model (forward bias model), and the simplified circuit is analysed accordingly.

The *equivalent circuit* is shown below, where the diode has now been replaced by a short circuit.




$$I = \frac{E}{R} = \frac{10}{10\text{ k}} = 1\text{ mA}$$

**Figure 3A.16**

Ohm's Law may be used to determine the current, *I*, as shown:

(ii) If the battery is reversed, the diode becomes *reverse biased*. In this case, the diode is replaced by the ideal diode model for reverse bias. Since the reverse biased ideal diode model is simply an *open circuit*, there is no current, i.e. $I = 0$.





**The Constant Voltage Drop Model**

A better model is to approximate the forward bias region with a vertical line that passes through some voltage called $e_{fd}$:

A model that takes into account the forward voltage drop

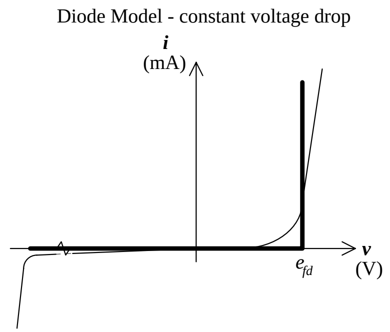



**Figure 3A.17**

This "constant voltage drop" model is better because it more closely approximates the characteristic in the forward bias region. The "voltage drop" is a model for the barrier voltage in the *p*-*n* junction. The model of the diode in this case is:

The constant voltage drop diode model

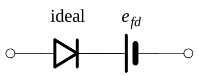



**Figure 3A.18**

### Example

(i) Find the current, *I*, in the circuit shown below, using the constant voltage drop model of the diode (assume $e_{fd} = 0.7\text{ V}$ ).

(ii) If the battery is reversed, what does the current become?

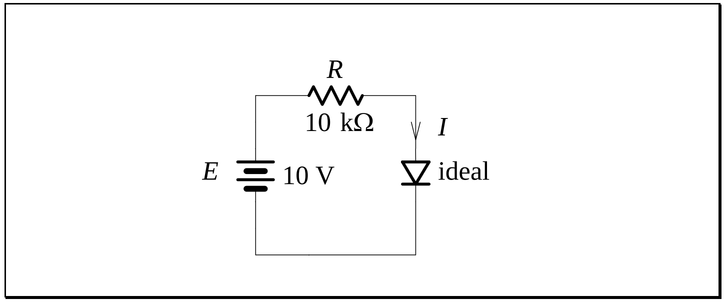
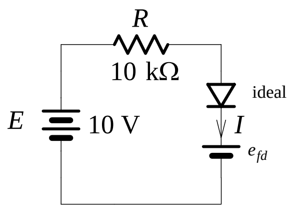



**Figure 3A.19**

(i) Analysis proceeds in exactly the same manner as the previous example, except that the constant voltage drop diode model is used instead. The diode is again forward biased, and so the equivalent circuit is shown below, along with the calculation for *I*.




$$I = \frac{E - e_{fd}}{R} = \frac{10 - 0.7}{10\text{k}} = 0.93\text{ mA}$$

**Figure 3A.20**

(ii) If the battery is reversed, the diode becomes *reverse biased*, resulting in no current, i.e. $I = 0$.





### The Piece-Wise Linear Model

An even better approximation to the diode characteristic is called a "piece-wise" linear model. It is made up of pieces, where each piece is a straight line:

A model that approximates the characteristic by using straight lines

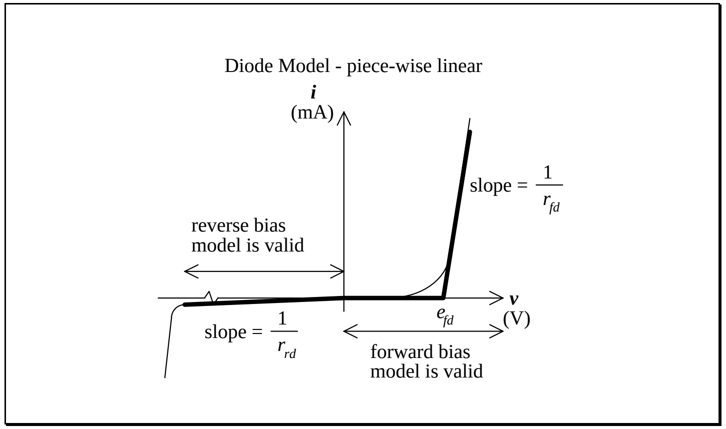



**Figure 3A.21**

For each section, we use a different diode model (one for the forward bias region and one for the reverse bias region):

The piece-wise linear diode model

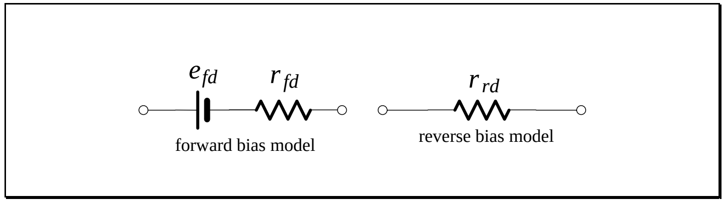



**Figure 3A.22**

Typical values for the resistances are $r_{fd} = 5\,\Omega$ and $r_{rd} > 10^9\,\Omega$.

Notice how we have done away with the ideal diode part of the model. This is because there is a separate equivalent circuit for the forward bias and reverse bias regions, so an ideal diode is not necessary (we apply one equivalent circuit or the other).

*You should verify, by using KVL, that these models actually give rise to the straight line characteristics shown in* Figure 3A.21.

**Example**

(i) Find the current, $I$, in the circuit shown below, using the piece-wise linear model of the diode (assume $e_{fd} = 0.7\text{ V}$, $r_{fd} = 5\,\Omega$ and $r_{rd} = \infty$).

(ii) If the battery is reversed, what does the current become?

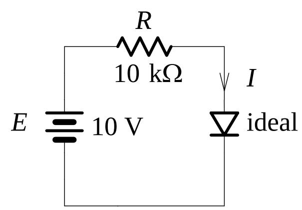
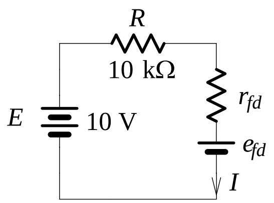



**Figure 3A.23**

(iii) Analysis proceeds in exactly the same manner as the previous example, except that the piece-wise linear diode model is used instead. The diode is again forward biased, and so the equivalent circuit is shown below, along with the calculation for $I$.




$$I = \frac{E - e_{fd}}{R + r_{fd}} = \frac{10 - 0.7}{10\text{k} + 5} = 0.9295\text{ mA}$$

**Figure 3A.24**

(iv) If the battery is reversed, the diode becomes *reverse biased*, and the diode is replaced by the piece-wise linear model. Since $r_{rd}$ is infinite, it acts as an open circuit, resulting in no current, i.e. $I = 0$.





**The Small Signal Model**

Suppose we know the diode voltage and current exactly. Would we still have a need for a linear diode model? *Yes*. Suppose the diode has a DC voltage and current. We may want to examine the behaviour of a circuit when we apply a signal (a small AC voltage) to it. In this case we are interested in small excursions of the voltage and current about some "DC operating point" of the diode. The best model in this instance is the following (the forward bias region is used as an example, but the method applies anywhere):

A model that approximates the characteristic by a tangent at a DC operating point

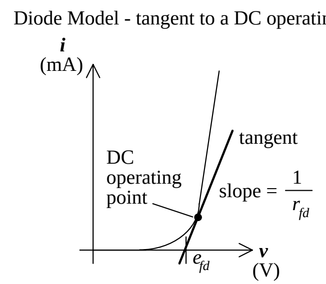



**Figure 3A.25**

We approximate the curved characteristic by the *tangent* that passes through the operating point. It is only valid for small variations in voltage or current. This is called the *small signal approximation*. A straight line is a good approximation to a curve if we don't venture too far.

A first look at the small signal approximation

The model we get in this case is exactly the same as in Figure 3A.22 except the values of $e_{fd}$ and $r_{fd}$ are different for each DC operating point.

Finally, to complete all our models, we can add a capacitance in parallel to model the forward and reverse capacitance described previously. We will not in general include the capacitance because it only becomes important at very high frequencies.

The capacitance of the diode is added last, but only used at high frequencies

For example, the piece-wise linear models become:

The piece-wise linear model for a diode that includes capacitance

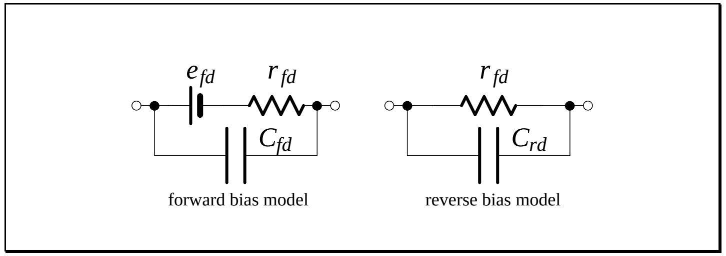



**Figure 3A.26**

## The Hall-effect Device

A Hall-effect device uses semi-conductors and the Lorentz Force Law

Suppose we have a doped semiconductor that has a current passing through it. Now imagine subjecting the semiconductor to a perpendicular magnetic field. *Show that the Lorentz Force Law says that the charge carriers will experience a force downwards regardless of their type (hole or electron)*:

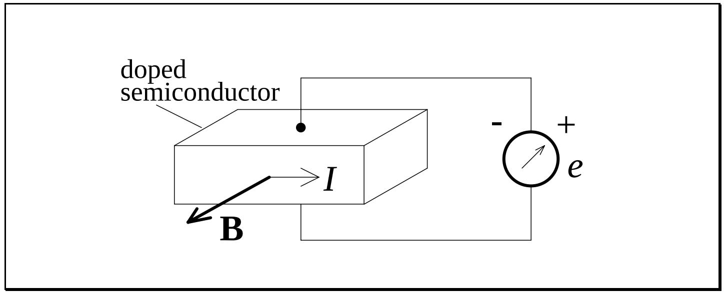



**Figure 3A.27**

An electric field is set up across the semiconductor. *Show the direction of this field for both types of charge carrier*. Its direction (and hence the potential difference across the semiconductor) is dependent upon the charge carrier. We can use this to measure the strength of magnetic fields or to determine the type of semiconductor (*p* or *n*).

and can be used to measure magnetic fields





## Breakdown Diodes

Some diodes are designed to operate in the breakdown region. It is usually a sharper transition than the forward bias characteristic, and the breakdown voltage is higher than the forward conduction voltage. There are two main types of breakdown.

Some diodes are designed to operate in the breakdown region

### Zener Breakdown

The electric field in the depletion layer of a *p*-*n* junction becomes so large that it rips covalent bonds apart, generating holes and electrons. The electrons will be accelerated into the *n*-type material and the holes into the *p*-type material. This constitutes a reverse current. Once the breakdown starts, large numbers of carriers can be produced with negligible increase in the junction voltage.

Zener breakdown is caused by a large internal electric field

### Avalanche Breakdown

If the minority carriers are swept across the depletion region of a *p*-*n* junction too fast, they can break the covalent bonds of atoms that they hit. New electron-hole pairs are generated, which may acquire sufficient energy to repeat the process. An avalanche starts.

Avalanche breakdown is caused by electrons with a large kinetic energy

## The Photodiode

In a photodiode, the *p*-*n* junction is very close to the surface of the crystal. The Ohmic contact with the surface material is so thin, it is transparent to light. Incident light (photons) can generate electron-hole pairs in the depletion layer (a process called photoionisation).

A photodiode is controlled by light

## The Light Emitting Diode (LED)

When a light-emitting diode is forward biased, electrons are able to recombine with holes within the device, releasing energy in the form of light (photons). The color of the light corresponds to the energy of the photons emitted, which is determined by the “energy gap” of the semiconductor. LEDs present many advantages over incandescent and compact fluorescent light sources including lower energy consumption, longer lifetime, improved robustness, smaller size,

An LED emits photons when forward biased faster switching, and greater durability and reliability. At the moment LEDs powerful enough for room lighting are relatively expensive and require more precise current and heat management than compact fluorescent lamp sources of comparable output.

LEDs are used in diverse applications. The compact size of LEDs has allowed new text and video displays and sensors to be developed, while their high switching rates are useful in advanced communications technology. Infrared LEDs are also used in the remote control units of many commercial products including televisions, DVD players, and other domestic appliances

## The Schottky Diode

A Schottky diode is a metal-semiconductor junction

A Schottky diode is the result of a metal-semiconductor junction. The Schottky diode is a much faster device than the general purpose silicon diode. There are three main reasons for this: 1) the junction used is a metal-semiconductor junction, which has less capacitance than a *p-n* junction, 2) often the semiconductor used is gallium arsenide (GaAs) because electron mobility is much higher, and 3) the device size is made extremely small. The result is a device that finds applications in high speed switching and decoupling operations.

## The Varactor Diode

This device is also known as a variable capacitance diode. It has a relatively large capacitance, brought about by a large junction area and narrow depletion region. The applied reverse voltage changes the length of the depletion region, which changes the capacitance. Thus, the device can be used in applications that rely on a voltage controlled capacitance. Applications include electronic tuning circuits used in communication circuits, and electronic filters.





## Summary

- Semiconductors are crystals which are insulators at low temperatures. Thermal energy in a pure (intrinsic) semiconductor can create an electron-hole pair, allowing conduction.

- Specific impurities are added to semiconductors in controlled amounts (a process called doping) to increase the number of charge carriers. A semiconductor doped with acceptors is rich in holes and is therefore called *p*-type. A semiconductor doped with donors is rich in electrons and is therefore called *n*-type.

- A *p-n* junction is formed in a semiconductor where the impurity type changes from *p* to *n*, while the lattice structure continues undisturbed. The *p-n* junction creates a depletion region which forms a potential barrier. The potential barrier can be increased or decreased with the application of an external voltage. The external voltage can therefore be used to change the conductivity of the semiconductor.

- A *p-n* junction forms a circuit element known as a diode. A diode’s characteristic is broken down into two regions – the forward bias region and the reverse bias region. In the reverse bias region, the diode is effectively an open circuit. In the forward bias region, and once the internal potential barrier is overcome, the diode can conduct.

- There are numerous circuit models for the diode. In general, the choice of diode model to be used is based on three main issues: the available diode data, the accuracy required, and the relative complexity of analysis involved. The most commonly used model is the constant voltage drop model.

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# Lecture 3B – Field Mapping

*The method of curvilinear squares. The coaxial cable. The two conductor transmission line.*

## The Method of Curvilinear Squares

There are various methods we can employ to map out a field. The method of curvilinear squares is based upon the plotting of lines of force and equipotentials, just like our original picture of fields. It is done by hand, and may be iterative. It is used to get an idea of what the field "looks" like and to get estimates of capacitance and inductance of mathematically difficult systems.

One method of field plotting is the method of curvilinear squares

You can conceivably obtain a field "plot" of a three dimensional (3D) field if you are prepared to model in 3D. e.g. construction of a 3D grid with wires representing lines of force and equipotentials.

On paper (the most convenient material) we are restricted to two dimensions (2D), so this method is normally based on 2D problems.

Field plotting is mainly used for 2D problems

Consider a 3D arrangement of conductors that have uniform cross-section, and are infinitely long. There are no field components in the longitudinal direction. (*Why?*) We only have to analyse the field by taking a cross-section. We have seen this before: the infinitely long conductor, the coaxial cable.

2D plots may be applied to 3D problems in certain cases

Consider the electrostatic field around a point charge:

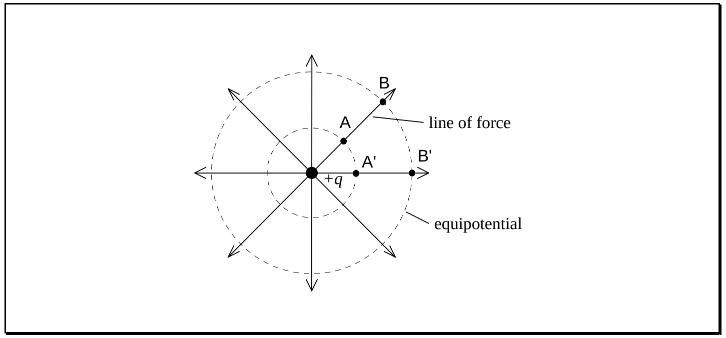


**Figure 3B.1**

The electric field at A or A', distance $R_A$ from the charge is:

$$\mathbf{E}_A = \frac{q}{4\pi\varepsilon_0 R_A^2}\hat{\mathbf{R}} \tag{3B.1}$$

The absolute potential around a point charge (revisited)

The potential (with respect to infinity) is:

$$\begin{aligned} V_A &= -\int_\infty^{R_A} \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^{R_A} -Edl \\ &= -\int_\infty^{R_A} EdR = \frac{-q}{4\pi\varepsilon_0}\int_\infty^{R_A}\frac{dR}{R^2} = \frac{q}{4\pi\varepsilon_0 R_A} \end{aligned} \tag{3B.2}$$

to illustrate the concept of "equipotential"

The potential is independent of where the point $A$ lies on the circle. It is only dependent on the distance from the charge. Hence the circle with radius $R_A$ is an equipotential.

Equipotentials and lines of force are always at right angles in electrostatics

Equipotentials are always at right angles to lines of force. Imagine a test charge being moved perpendicular to the direction of the field at all times. Then:

$$\int_A^B \mathbf{E}\cdot d\mathbf{l} = \int_A^B E\cos\theta dl = \int_A^B E\cos 90° dl = 0 \tag{3B.3}$$

The surface of a metal with a static charge is an equipotential, since the tangential part of **E** is zero on the surface. (If **E** were not zero, then charges would redistribute themselves on the surface until there was no force on them – a condition which means the tangential part of **E** is zero).

A field plot is a plot of equipotentials and lines of force

We can now consider a field plot to be composed of two families of lines: one representing lines of force (or equivalently, lines defining tubes of flux); the other representing equipotentials. We will always know where to draw some of the equipotentials: at the surface of conductors.

The field around a point charge (drawn in Figure 3B.1) can be considered as a cross-section of the field around an infinitely long line charge, as far as the field plot is concerned (the previous equations do not apply of course).





In Figure 3B.1, the element:

A curvilinear square shown pictorially

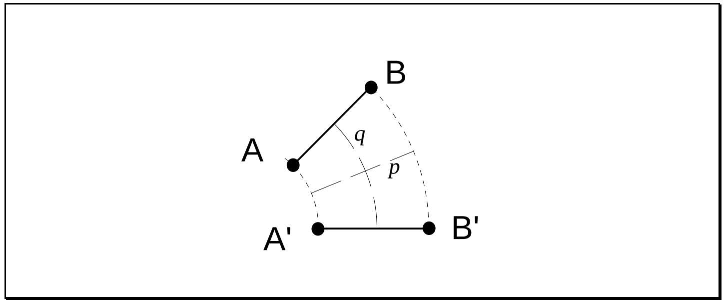


**Figure 3B.2**

and defined mathematically

is called a *curvilinear square* if $p = q$. A curvilinear square is a shape with four sides that tends to yield true squares as it is subdivided into smaller and smaller areas by successive halving of the equipotential interval and the flux per tube.

Drawing curvilinear squares is a "trial and error" method

We can draw field lines to satisfy the requirement that the density of lines is proportional to the field. We can then draw in equipotentials to obtain curvilinear squares. We can also *not* obtain curvilinear squares, which means the field lines are wrong. The whole process starts again by modifying the field lines to obtain curvilinear squares (if the plot is done in pencil). In other words, we proceed in an iterative fashion (if we knew what the field looked like to begin with, there would be no need to use this method, would there?).

This method of field plotting is very useful for irregular shapes and arrangements of conductors.

Consider the electric field shown below:

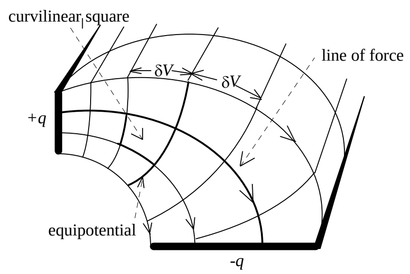


**Figure 3B.3**

The potential difference between the two conductors is $V$ volts. The LHS conductor has a distributed charge $+q$ and the RHS has $-q$. This is a bad plot. *Why*? Because the last equipotential converges onto another equipotential. The plot will have to be corrected. *Correct the above field plot. Hint: the field lines are wrong too.*

Why do a field plot? One reason is to obtain an estimate of capacitance (per unit length)

Once we get the plot visually right (the curvilinear requirement is met), we may wish to determine the capacitance per unit length between the two conductors, using the field plot.

We know that the capacitance between two conductors is given by:

$$C = \frac{q}{V} \tag{3B.4}$$

We also know from Gauss' Law around one of the conductors:

$$q = \psi \tag{3B.5}$$

where $\psi$ is the flux emanating from the conductor.





We could then say:

Capacitance defined in terms of flux and potential

$$C = \frac{\psi}{V} \tag{3B.6}$$

To calculate capacitance using this formula, we should first consider an isolated curvilinear cube:

A curvilinear cube has flux streaming *through* it, and potential *across* it

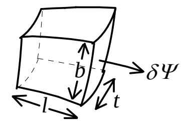


**Figure 3B.4**

It has a small amount of flux streaming through it, and a small voltage across it. It therefore contributes to the capacitance in some way. If the curvilinear cube is very small, then the flux density **D** may be assumed uniform across the face of the cube so that:

A curvilinear cube is a small capacitor

The flux streaming through a curvilinear cube

$$\begin{aligned} \delta\Psi &\approx \mathbf{D} \cdot \delta\mathbf{A} \\ &= \varepsilon E b t \end{aligned} \tag{3B.7}$$

We can approximate the electric field magnitude $E$ by calculating the small potential that exists across the curvilinear cube:

and the potential across it

$$\begin{aligned} \delta V &= -\int_l \mathbf{E} \cdot d\mathbf{l} \\ &= El \end{aligned}$$

$$E = \frac{\delta V}{l} \tag{3B.8}$$

Therefore, the amount of flux streaming through the cube may be expressed as:

$$\delta\Psi \approx \varepsilon \frac{\delta V}{l} bt \tag{3B.9}$$

Also, if the cube is small, $l \approx b$ and the flux is given by:

$$\delta\Psi \approx \varepsilon \delta V t \tag{3B.10}$$

The total amount of flux streaming from one of the conductors is obtained by adding up all the small amounts of flux streaming through each flux tube:

The total flux streams through all the curvilinear cubes that are in parallel

$$\psi = \sum_{n_p} \delta\Psi \tag{3B.11}$$

where $n_p$ is the number of flux tubes in parallel (number of curvilinear squares in *parallel*).

The total potential between the two conductors is obtained by adding up all the small amounts of potential between each equipotential, in going from one conductor to the other:

The total voltage is across all the curvilinear cubes that are in series

$$V = \sum_{n_s} \delta V \tag{3B.12}$$

where $n_s$ is the number of equipotentials minus one (number of curvilinear squares in *series*).

We can now determine the capacitance of the structure in this way:

The capacitance using curvilinear squares

$$C = \frac{\psi}{V} = \frac{\sum_{n_p} \delta\Psi}{\sum_{n_s} \delta V} = \frac{\sum_{n_p} \varepsilon \delta V t}{\sum_{n_s} \delta V} \tag{3B.13}$$





But since $\delta V$ is the same value for each curvilinear square, we have:

$$\sum_{n_p} \varepsilon \delta V t = \varepsilon \delta V t n_p$$
$$\sum_{n_s} \delta V = \delta V n_s \tag{3B.14}$$

We can now define the capacitance per unit length of the two conductors. This is all we can calculate, since the capacitance of infinitely long conductors is infinite. Our answer may be applied to very long conductors with a small error.

The capacitance per unit length using curvilinear squares

$$\frac{C}{t} = \varepsilon \frac{n_p}{n_s} \quad \text{Fm}^{-1} \tag{3B.15}$$

When field plotting, we exploit symmetry

**Example – Rectangular conductor between two earth planes**

Consider a rectangular conductor between two earth plates. Due to the symmetry of the arrangement, only 4 of the field needs to be plotted:

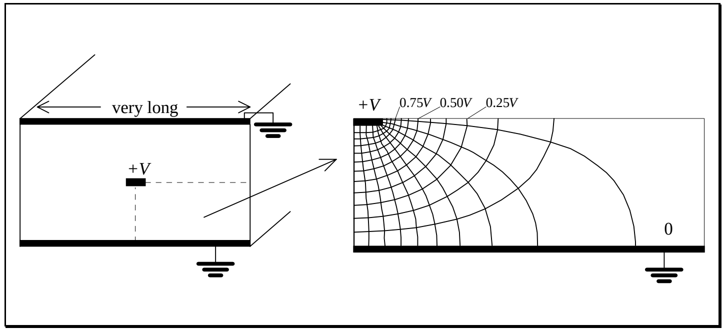


**Figure 3B.5**

The capacitance per unit length in this case is:

$$\frac{C}{t} = \varepsilon \frac{n_p}{n_s} = \varepsilon \frac{4 \times 9.5}{12} = \frac{19}{6} \varepsilon \ \text{Fm}^{-1}$$





**Example – Cylindrical conductor inside metal duct**

Due to the symmetry of the arrangement, only 1/8 of the field needs to be plotted. The surfaces of the inner conductor and of the duct are assumed to be perfect equipotentials.

A mathematically difficult problem made easy

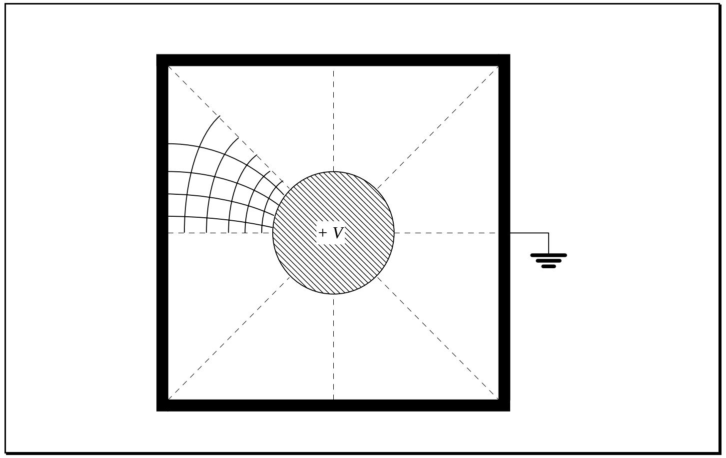


**Figure 3B.6**

*Calculate the capacitance per unit length for the above arrangement.*

## The Coaxial Cable

A long co-axial cable is approximated by one of infinite length – so we can plot the field

A long coaxial cable has a simple symmetry and can be approximated by an infinitely long cable. We have seen it before in the problems. You can derive the formula for capacitance per unit length analytically using the method of curvilinear squares and compare it with that obtained by finding the electric flux density, electric field, voltage and then capacitance per unit length as done previously.

A mathematically easy problem is used to verify the method of curvilinear squares

The method we use is identical to that used to determine the dielectric resistance of a co-axial cable. The dielectric may be assumed to consist of a very large number of concentric tubes, each with a tiny thickness:

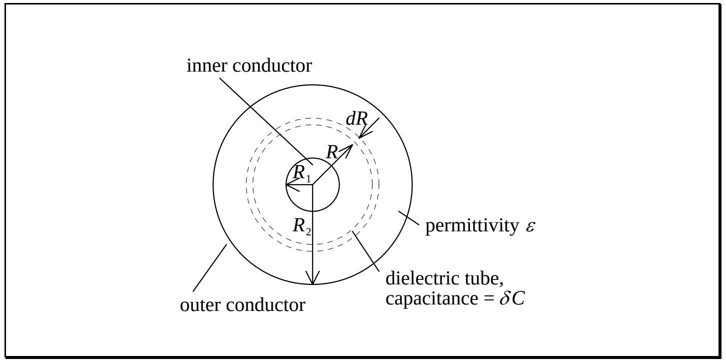


**Figure 3B.7**

For the dielectric tube shown:

The capacitance of a cylindrical tube with finite thickness

$$\delta C = \frac{\psi}{\delta V} = \frac{\varepsilon E \cdot 2\pi R l}{\delta V} \approx \frac{\varepsilon 2\pi R l}{\delta R} \tag{3B.16}$$

where $l$ = length of the cable.





In the limit, for an infinitesimally thin flux tube, the capacitance is:

The capacitance of a cylindrical tube with infinitesimal thickness

$$dC = \frac{\varepsilon 2\pi R l}{dR} \tag{3B.17}$$

As all the tubes of flux are concentric, the capacitances $dC$ are in series and:

The tubes are added in series to give the total capacitance

$$\frac{1}{C} = \int \frac{1}{dC} = \int_{R_1}^{R_2} \frac{dR}{\varepsilon 2\pi R l} = \frac{1}{2\pi\varepsilon l}\int_{R_1}^{R_2} \frac{dR}{R} = ? \tag{3B.18}$$

*Complete the analysis to determine a formula for* C.

## The Two Conductor Transmission Line

To calculate the capacitance between two infinitely long conductors, we assume an electrostatic situation – we ignore any current in the conductors and analyse the effect of the charge that has drifted to and remained on the conductor surface. Since we assume a static state of the charge, the surface of the conductor is an equipotential. We then model the surface charge as a line charge at the centre of the conductor:

A surface charge is modelled by a line charge

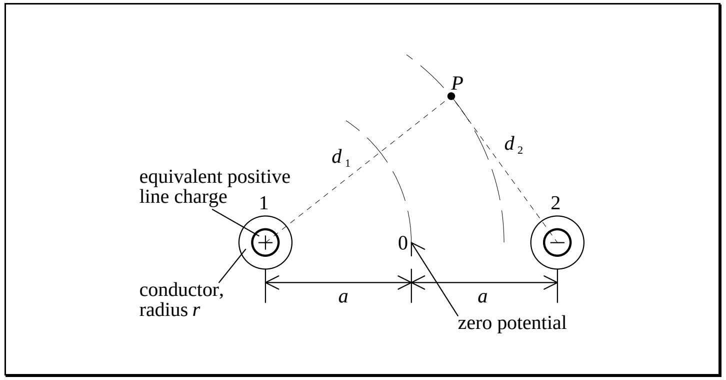


**Figure 3B.8**

The magnitude of the electric field at radius $x$ due to the positive line charge is:

$$E = \frac{\lambda}{2\pi\varepsilon x} \tag{3B.19}$$

where $\lambda$ = charge / unit length. The electric potential at point $P$ is:

The potential at a point due to one conductor

$$V_P = -\int_0^P \mathbf{E} \cdot d\mathbf{l} = -\int_a^{d_1} \frac{\lambda}{2\pi\varepsilon x} dx = \frac{\lambda}{2\pi\varepsilon} \ln \frac{a}{d_1} \tag{3B.20}$$

Note that the point of zero potential is arbitrarily taken to be midway between the conductors. (It does not matter where we define "zero" potential, since the only meaningful concept is potential *difference*).





By superposition (assuming a linear medium, such as air), due to both line charges, we get:

$$V_P = \frac{\lambda}{2\pi\varepsilon} \ln \frac{d_2}{d_1} \tag{3B.21}$$

The potential at a point due to both conductors

as the total potential at point $P$.

The voltage on the surface of the positive conductor (radius $r$) is similarly given by:

$$V_1 \approx \frac{\lambda}{2\pi\varepsilon} \ln \frac{2a}{r} \quad (\text{if } 2a >> r) \tag{3B.22}$$

The approximate potential at the surface of a conductor

The capacitance per unit length between conductor 1 and the zero potential line is therefore:

$$\frac{C_{10}}{l} = \frac{2\pi\varepsilon}{\ln(2a/r)} \tag{3B.23}$$

The capacitance per unit length between one conductor and zero potential

By symmetry, the capacitance per unit length between conductors 1 and 2 is:

$$\frac{C_{12}}{l} = \frac{1}{2}\frac{C_{10}}{l} = \frac{\pi\varepsilon}{\ln(2a/r)} \tag{3B.24}$$

The capacitance per unit length between the two conductors

(i.e. $C_{10} = C_{20}$ and in series).

The field between conductors of different radii is handled in the same way as the transmission line – an equivalent line charge is located somewhere inside the conductor so that the surface of the conductor is an equipotential:

An equipotential can be used as the surface of a conductor

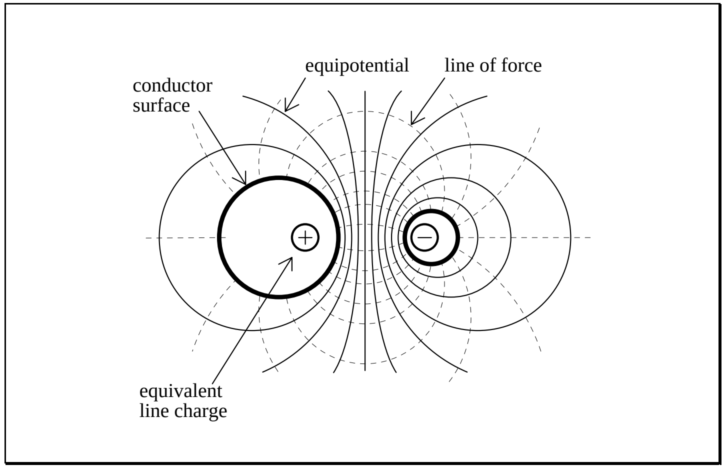


**Figure 3B.9**





## Summary

- A field plot is a plot of equipotentials and lines of force. Two dimensional plots are normally done on paper or a computer.
- Field plots use the concept of a curvilinear square – a shape which has curved sides of roughly equal length.
- Field plots can be used to estimate the capacitance per unit length of irregular shapes and arrangements of conductors.

## References


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# Lecture 4A – Diode Circuits

*The peak detector. The clamp circuit. The clipping circuit.*

## The Peak Detector

Consider the following circuit:

A peak detector circuit

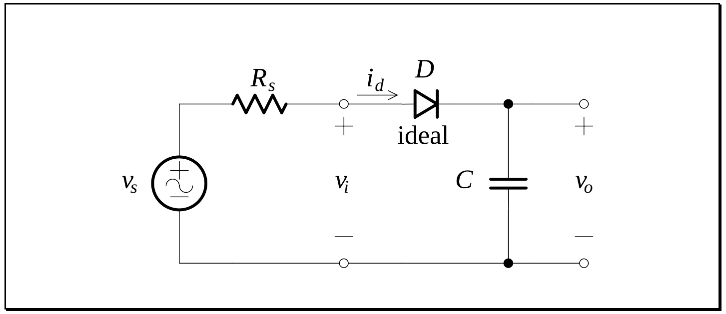



**Figure 4A.1**

Normal circuit analysis cannot be used with diodes

The state of the diode will affect the analysis of the circuit. Since there are only two ways in which an ideal diode can operate ("on" or "off"), we will assume that the diode is in some state initially. Once a diode is assumed to be "on" or "off", an analysis of the circuit can be carried out.

Circuit analysis starts with an assumption – which is later checked

After the analysis, we will check our original assumption to see that it is valid. If it is, then the analysis is complete, otherwise we assume the opposite state for the diode, and carry out another analysis. For example, after assuming the diode to be "on", we may find that the current in the diode goes from cathode to anode – which is impossible. The initial assumption of the diode being "on" must therefore be wrong. We should start the analysis again, but this time assume that the diode is "off".

We always assume that diodes are initially "off"

In analysing circuits with diodes, we will *always* initially assume that a diode is in the "off" state. After carrying out an analysis, we should check that the diode is indeed in the "off" state. This condition corresponds to the diode being reverse biased. In other words, the diode voltage, $v_d$, should be negative. If it is not, then the diode is really "on", and we have to repeat the analysis.

Let’s first look at an isolated capacitor subjected to a sinusoidal voltage:

The current and voltage relationship in a capacitor

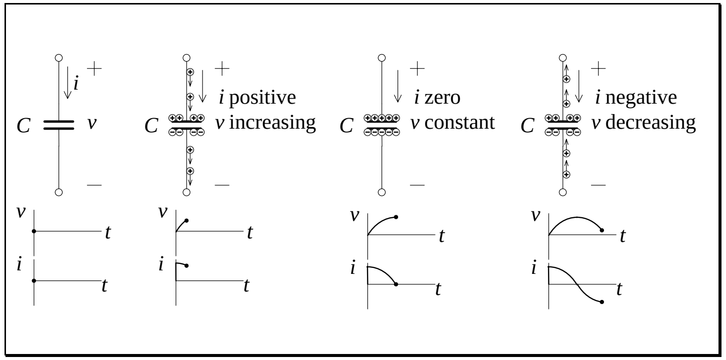



**Figure 4A.2**

We know for a capacitor that:

$$q = Cv \tag{4A.1}$$

When a capacitor stores charge, it has positive charge on one plate, and negative on the other. An electric field therefore exists between the plates of the capacitor. To move a positive charge from the negative plate to the positive plate through the electric field means doing work. The voltage across a capacitor is the work done per unit charge in doing this. For each voltage, there corresponds a unique proportional charge. The proportionality constant is called the capacitance.

Storing charge in a capacitor produces a voltage

Initially the capacitor holds no charge, so the voltage across it is zero. (This doesn’t mean it is a short circuit, it just means there is no electric field to oppose moving a charge from one plate to the other).

If there is no stored charge in a capacitor, there is no voltage

When the voltage across the capacitor is increased, the charge increases in proportion. We are “putting positive charge onto the positive plate, and removing it from the negative plate”. With the sign convention as in Figure 4A.2, this implies a positive current.

If the voltage across a capacitor is not changing, then the charge it is storing cannot be changing either. Therefore, the current must be zero.





When we decrease the voltage across a capacitor, we have to decrease the amount of stored charge. This means we must "remove positive charge from the positive plate, and put it on the negative plate". This means negative current using our sign convention.

The above reasoning is summed up by differentiating Eq. (4A.1) with respect to time:

The current “through” a capacitor depends only on the rate of change of voltage with respect to time

$$i = C\frac{dv}{dt} \tag{4A.2}$$

Now consider our peak detector circuit again. Notice that the output is taken across the capacitor, so it is this voltage that we are interested in. We will assume that the source is a sine wave (not a cosine wave). We also assume, as always, that the diode is “off” initially. This means there is no current in the circuit and KVL around the loop gives:

$$v_s - v_d - v_C = 0 \tag{4A.3}$$

But since there is no charge on the capacitor initially:

$$\begin{aligned} v_s - v_d &= 0 \\ v_d &= v_s \end{aligned} \tag{4A.4}$$

This means the source voltage appears directly across the diode. But the source voltage is a sine wave, and it is positive initially. We can never have a positive voltage across an ideal diode – its characteristic does not allow it. We must have made a wrong assumption – the diode must initially be in the “on” state.

A positive voltage across an ideal diode means a wrong assumption - start again

With the diode “on”, KVL gives:

$$\begin{aligned} v_s - R_s i_d - v_C &= 0 \\ v_C &= v_s - R_s i_d \end{aligned} \tag{4A.5}$$

If the source resistance is small, then we may say:

$$v_C \approx v_s \tag{4A.6}$$

Therefore, initially, the output of the peak detector equals the input. The capacitor will have the same voltage as the source until it reaches its peak.

A negative current through an ideal diode means a wrong assumption - start again

As soon as the source voltage tries to reduce the capacitor voltage, we know that the current in the capacitor must be negative (with respect to the defined current direction). Since the diode blocks current in this direction, there will be no current. After the capacitor voltage reaches the peak of the source voltage, the diode will not allow reverse current, and will be reverse biased when the source voltage decreases from its peak value.

If the diode is "off", then the capacitor cannot discharge, so its voltage will be:

$$v_C \approx \hat{v}_s \tag{4A.7}$$

We analysed the circuit before when the diode was "off". In this case, Eq. (4A.3) gives for the diode reverse bias voltage:

$$v_d = v_s - v_C \approx v_s - \hat{v}_s \tag{4A.8}$$

From this equation, we can see that the diode voltage will always be negative or just on zero. The diode will therefore remain reverse biased for all time, so the capacitor will retain its voltage for all time.





A graph of the various voltages in the peak detector is shown below:

The peak detector output voltage

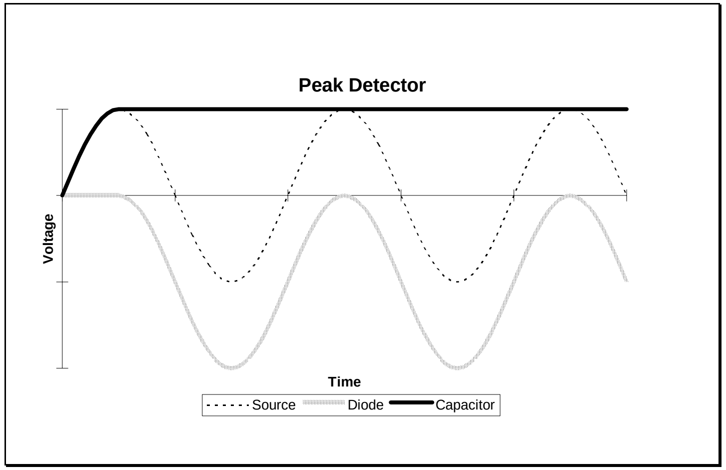



**Figure 4A.3**

The diode will not conduct again until the source voltage changes so as to exceed the capacitor voltage.

Loading the peak detector output

The steady-state output of the peak detector is obviously DC. What would happen if we try to use the peak detector as a DC voltage source? Consider the following circuit, which is just a peak detector with a load resistor attached to the output:

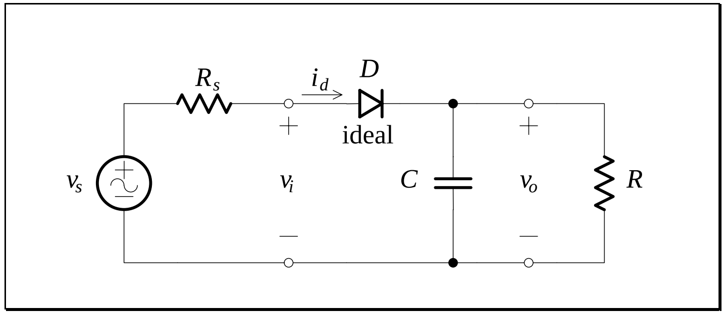



**Figure 4A.4**

Assume there is no charge on the capacitor initially. The circuit will behave exactly as before, and the diode will be in the "off" state at the peak of the source voltage. With the diode "off" the circuit looks like:

A load on a peak detector output now provides a path for capacitor discharge current

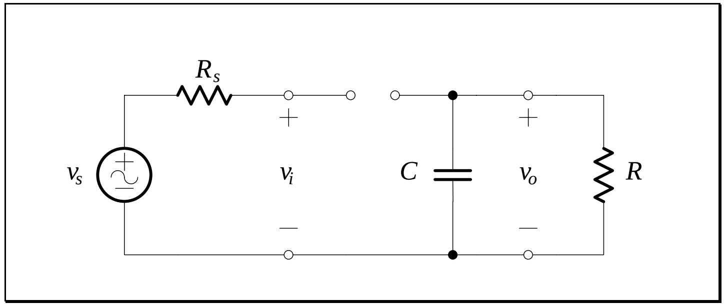



**Figure 4A.5**

Writing KCL at the output node gives:

$$C\frac{dv_o}{dt}+\frac{v_o}{R}=0 \tag{4A.9}$$

Rearranging and integrating with respect to time, we get:

$$\begin{aligned}\frac{1}{v_o}\frac{dv_o}{dt}&=\frac{-1}{RC}\\ \int_0^t\frac{1}{v_o}\frac{dv_o}{dt}dt&=\int_0^t\frac{-1}{RC}dt\end{aligned} \tag{4A.10}$$





Performing the integral and rearranging, we get an expression for the voltage across the load:

$$\begin{aligned}\ln\left[\frac{v_o(t)}{v_o(0)}\right]&=\frac{-t}{RC}\\ \frac{v_o(t)}{v_o(0)}&=e^{-t/RC}\\ v_o(t)&=v_o(0)e^{-t/RC}\\ v_o(t)&=\hat{V}_Ce^{-t/\tau}\end{aligned} \tag{4A.11}$$

We have assumed (arbitrarily) that $t=0$ is the instant the diode switches "off". Therefore, when the diode is "off", the voltage across the load experiences an exponential decay. The *time constant*, $\tau = RC$, is determined by the capacitor and the resistor. The larger the value of capacitance and resistance, the slower the decay.

The output voltage experiences exponential decay

The voltage will continue to decay until the input voltage has a higher value than the load voltage, at which point the diode turns "on". This will charge the capacitor up to the peak value of the input voltage again. A cycle will then be established.

until the next charging cycle

A graph of the voltages and the source current is shown below:

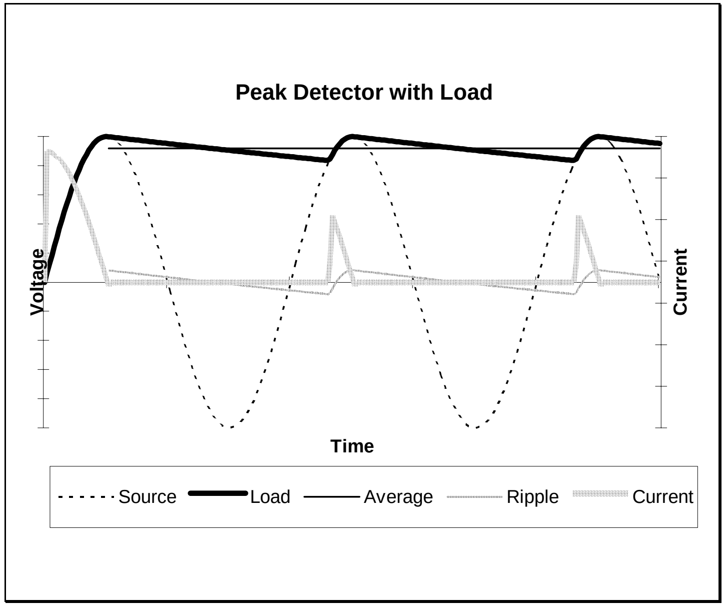



**Figure 4A.6**

If the time constant $\tau$ is large, then the exponential term in Eq. (4A.11) can be approximated by a linear term:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$
$$\approx 1 + x \quad \text{if } x \text{ is small}$$
$$v_o(t) \approx \hat{V}_C\left(1 - \frac{t}{RC}\right) \tag{4A.12}$$

For slow exponential decay, a straight line is a good approximation

The output, when the diode is "off", therefore looks like a straight line.

Since the discharge time is much larger than the charging time, we can approximate the discharge time by the period of the source, $T$.





The peak-to-peak ripple (AC) and average (DC) parts of the voltage are then given by:

$$V_r = \hat{v}_o - \hat{v}_o\left(1 - T/RC\right)$$
$$= \frac{\hat{v}_o T}{RC} = \frac{\hat{v}_o}{fRC}$$
$$V_{DC} = \hat{v}_o - \tfrac{1}{2}V_R$$
$$= \hat{v}_o - \frac{\hat{v}_o}{2\,fRC} \tag{4A.13}$$

The approximate DC voltage produced by a loaded peak detector

To decrease the ripple we choose large values for $R$ and $C$, and if we can, $f$.

To observe the ripple, we can pass the output voltage to a capacitively coupled load (e.g. a DSO on AC coupling).

## The Clamp Circuit

Consider the following circuit:

A positive clamp circuit

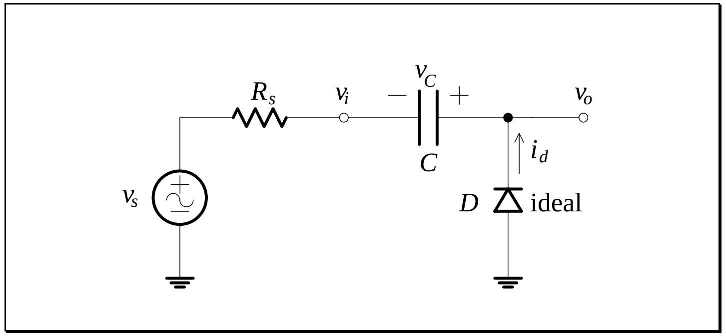



**Figure 4A.7**

Assume initially the diode is off. The source voltage is assumed to rise from zero – it is a sine wave. In addition, assume that $R_s$ is small and therefore $v_i \approx v_S$.

The diode will remain reverse biased until the source goes negative, since KVL around the loop gives:

$$\begin{aligned} v_s + v_d &= 0 \\ v_d &= -v_s \end{aligned} \tag{4A.14}$$

(Remember initially that the voltage across the capacitor is 0 V – it holds no charge. Also, there is no voltage across the source resistance, since the diode is like an open circuit – there is no current).

Our assumption that the diode is off breaks down when the source voltage goes negative, since then a positive voltage exists across the diode. This condition must not happen, so our assumption is wrong. The diode must be forward biased and also conducting current when this occurs.

The current charges the capacitor, so that the voltage defined in Figure 4A.7 has a positive value. Doing KVL around the loop gives for the capacitor voltage:

$$v_C = -v_i \tag{4A.15}$$

(Remember that the diode is ideal, so it has no voltage drop across it when conducting. Also remember that at this stage the input voltage is negative, so that the capacitor voltage, as defined by the above equation, will be a positive number).

The diode prevents the capacitor from discharging

When the source reaches its negative peak, the current will reach zero. *Why?* At this point, the current would like to reverse direction. The source voltage is trying to decrease in *magnitude*, which means the charge stored on the capacitor will have to decrease. This can only be achieved by a current that draws positive charge off the positively charged plate – in effect, a current in the opposite direction to that shown in Figure 4A.7. This also corresponds to wanting a negative current in the capacitor, as explained previously. Since current cannot go in this direction – it is prevented by the diode – there will be





no current and the diode may be considered off. With the diode off, the capacitor cannot discharge, and the voltage across it will be:

$$v_C = \hat{v}_i \tag{4A.16}$$

This is only true for a symmetric waveform. In general, the capacitor will charge to the magnitude of the negative peak of the waveform.

KVL around the loop then gives:

$$v_o = v_i + v_C = v_i + \hat{v}_i \tag{4A.17}$$

The output voltage is a shifted version of the input voltage – it is alternating but unipolar

The output is seen to be shifted by a DC voltage equal to the magnitude of the negative peak of the input voltage. The output voltage is said to have its lowest point *clamped* to zero – it cannot go below zero. The circuit is therefore known as a *positive clamp* circuit. Since the output is always positive, the diode is always reverse biased and will not conduct again. *Confirm this by doing KVL around the loop.*

A graph of the various voltages is shown below:

The output voltage of the positive clamp circuit

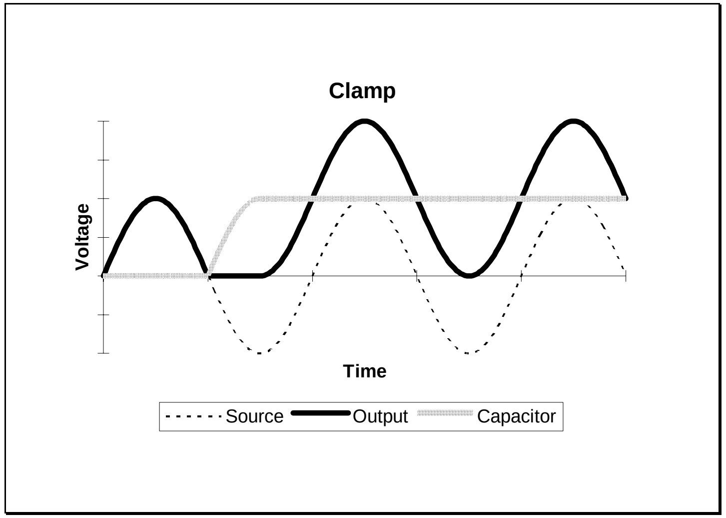



**Figure 4A.8**

## The Clipping Circuit

Consider the following circuit:

A clipping circuit

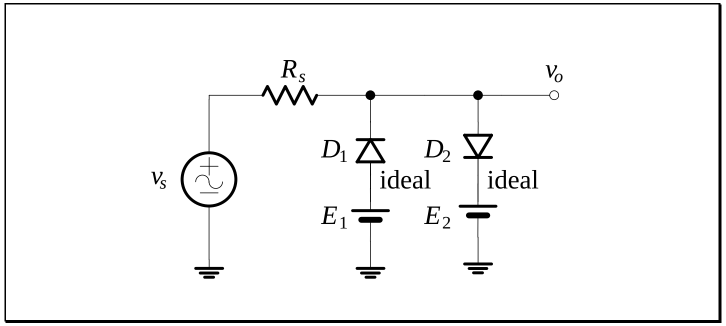



**Figure 4A.9**

The circuit works very simply. Assume both diodes are off. KVL then gives:

$$v_o = v_s \tag{4A.18}$$

If the output voltage is less than $E_1$, then diode $D_1$cannot be reversed bias, so it will conduct. This limits or clamps the output voltage to $E_1$:

$$v_o = E_1 \quad \text{for } v_s < E_1 \tag{4A.19}$$

If the output voltage is more than $E_2$ then diode $D_2$ cannot be reversed bias, and it turns on, limiting the output voltage to $E_2$:

$$v_o = E_2 \quad \text{for } v_s > E_2 \tag{4A.20}$$





A graph of the output is shown below:

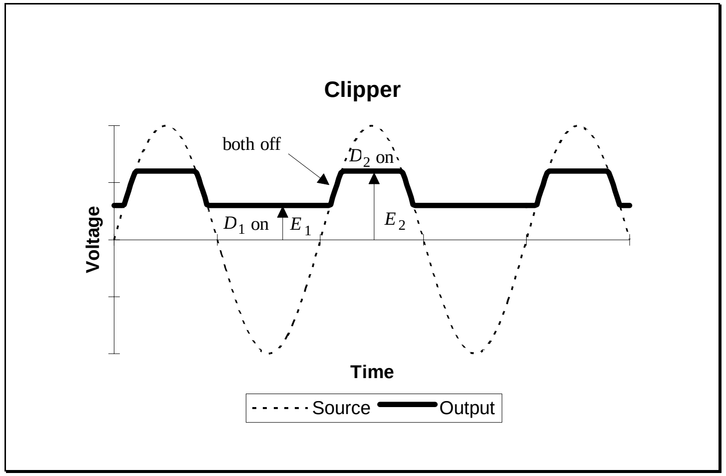



**Figure 4A.10**

Limiting can also be achieved by exploiting the breakdown voltage of a Zener diode.

### Graphical Analysis of Clipping Circuit

Consider the following circuit which clips at one level:

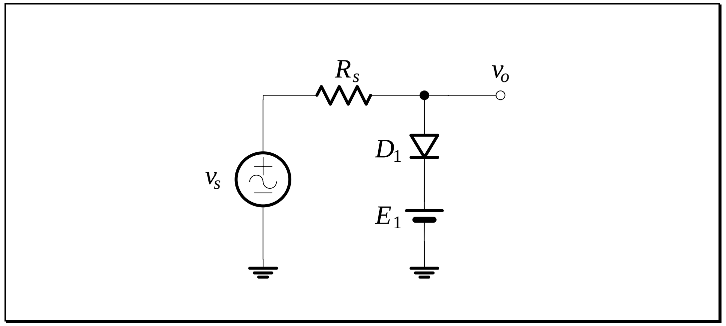



**Figure 4A.11**

Analysis is performed using a diode model

We would like to consider the effect of a real diode and examine the output waveform for any particular input waveform. To do this we will use a piece-wise linear model for the diode and draw the graph of the circuit's transfer function (i.e. a graph of output voltage versus input voltage).

First, we replace the diode with its model for the two cases of forward and reverse biased:

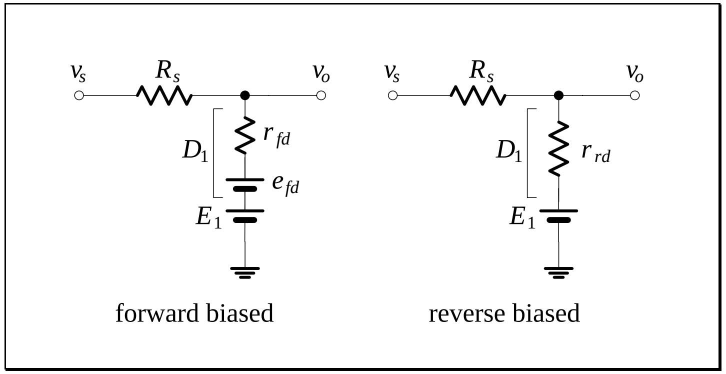



**Figure 4A.12**

*Show that the diode conducts when:*

$$v_s \geq E_1 + e_{fd} \tag{4A.21}$$

*When the diode conducts, show that analysis of the forward biased equivalent circuit in Figure 4A.12 gives:*

$$v_o = \frac{r_{fd}}{r_{fd} + R_s} v_s + \frac{R_s}{r_{fd} + R_s}\left(e_{fd} + E_1\right) \tag{4A.22}$$

*Hint: use superposition (since there are two independent sources) and the voltage divider rule.*





*Show that for the reverse biased case:*

$$v_o \approx v_s \tag{4A.23}$$

*Hint: the resistance $r_{rd}$ can be assumed to be much larger than $R_s$.*

The transfer characteristic allows analysis of any input waveform

These two equations, corresponding to the two different states of the diode, give the relationship between the output voltage and source voltage. They are valid only in the region for which the diode model is valid, as determined by Eq. (4A.21). Graphing these two equations, in their appropriate regions, gives the transfer characteristic for the circuit:

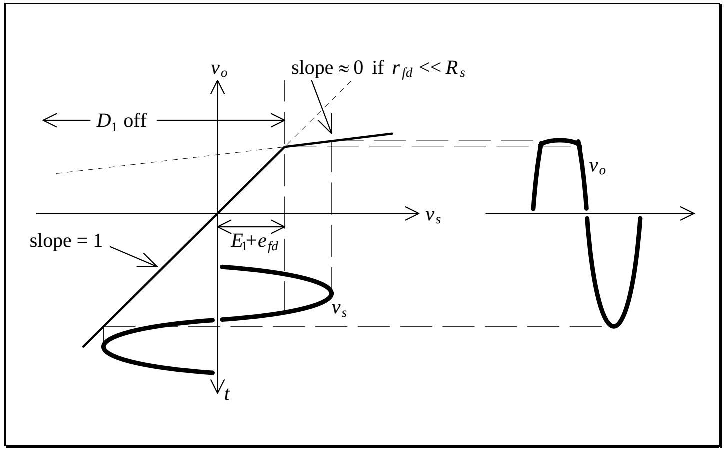



**Figure 4A.13**

The effect this circuit has on a sine wave is shown.

## Summary

- A peak detector charges a capacitor to the peak value of the input voltage. In normal operation, a load resistor normally exists on the output of the peak detector. In this case, the capacitor voltage decays exponentially until the input charges the capacitor again. This causes *ripple* in the output voltage.
- A clamp circuit maintains the integrity (shape) of the input waveform, but the DC level is shifted up or down, depending on the diode direction.
- Clipping or limiting circuits are used to endure that a voltage output is maintained within certain limits.

## References

Sedra, A. and Smith, K.: *Microelectronic Circuits*, Saunders College Publishing, New York, 1991.

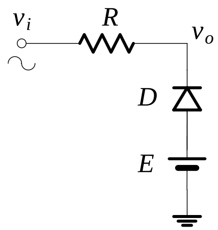




# Problems

**1.**

Consider the clipping circuit shown opposite. Draw equivalent circuits for both forward and reverse biasing of the diode.




Given that:

$$\hat{v}_i = 2(E - e_{fd}), \quad r_{fd} = \frac{R}{5}, \quad r_{rd} >> R,$$

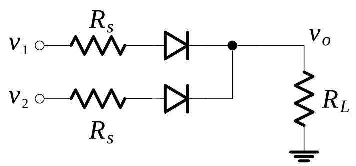
and that $E = 5\text{ V}$ and $e_{fd} = 0.7\text{ V}$, use a graphical method to obtain $v_o$.

**2.**

Obtain expressions for the output voltage $v_o$, if:

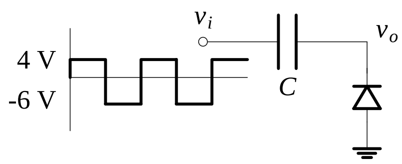
(i) $v_1 = v_2 = V$ (DC)

(ii) $v_1 = V$, $v_2 = 0$




Assume a constant voltage drop model, with $e_{fd} = 0.7\text{ V}$.

**3.**




Sketch $v_o$. Indicate peak values.

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# 4A.18

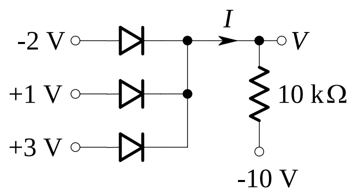
**4.**

Determine $V$ and $I$ in the following circuits, when:

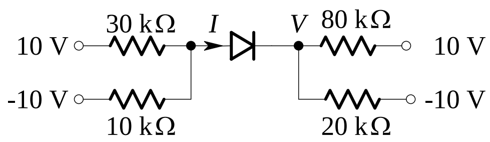
a) The diodes are assumed to be ideal.

b) The diodes are modelled with a constant voltage drop model with $e_{fd} = 0.7\text{ V}$.

(i)




(ii)



# Lecture 4B – Magnetic Circuits

*The magnetic circuit. Magnetic and electric equivalent circuits. DC excitation. AC excitation. Characteristics. Determining $\mathcal{F}$ given $\phi$. Determining $\phi$ given $\mathcal{F}$ (load line). Equivalent circuit of a permanent magnet.*

## The Magnetic Circuit

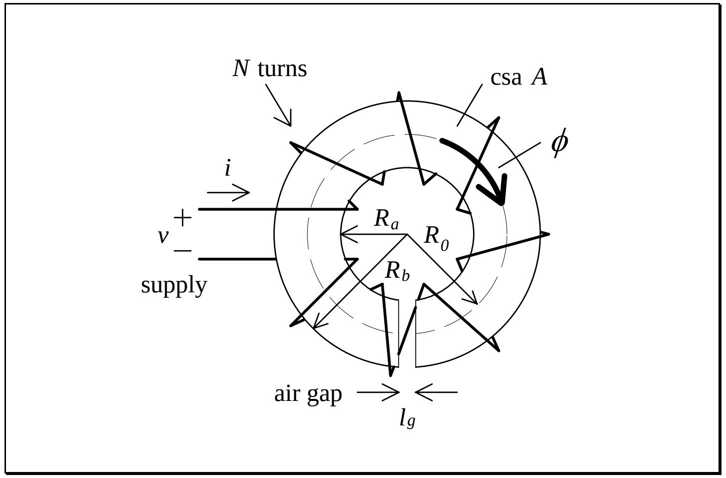



**Figure 4B.1**

The magnetic field in a "thin" toroid is uniform

Consider a toroid with a core of ferromagnetic material. A small gap is made in the core. We know from previous analysis and by demonstration, that the magnetic field inside a toroid is fairly uniform. No magnetic field lies outside the toroid. For this case, the flux path is defined almost exactly.

An analogy between magnetic and electric circuits

The ferromagnetic material can be considered a good "conductor" of flux, just like a metal wire is a good conductor of charge. The surrounding air, because of its low permeability, acts like an insulator to the flux, just like ordinary insulation around a metal wire. Since the flux path is well defined, and since the magnetic field is assumed to be uniform, Ampère's Law will reduce to a simple summation.

Ignore fringing of the flux in "small" air gaps

If the air gap is small, there will not be much fringing of the magnetic field, and the cross section of the air that the flux passes through will be approximately equal to the core cross section.

Let the cross sectional area of the toroid's core be $A$. The mean length of the core will be defined as the circumference of a circle with a radius the average of the inner and outer radii of the core:

$$l = 2\pi R_0 = 2\pi \frac{R_a + R_b}{2} \tag{4B.1}$$

Ampère's Law around the magnetic circuit gives:

Ampère's Law for a "thin" toroid

$$\oint_l \mathbf{H} \cdot d\mathbf{l} = Ni$$

$$\oint_l H dl = \sum_l Hl = \sum_l U = Ni \tag{4B.2}$$

The integral can be simplified to a summation, since the field **H** is in the same direction as the path **l**. This is a direct result of having a ferromagnetic material to direct the flux through a well defined path. In all cases we will let the subscript $i$ mean iron (or any ferromagnetic material) and subscript $g$ mean gap (in air). Ampère's Law written explicitly then gives:

turns into a simple summation

$$\begin{aligned} Ni &= H_i l_i + H_g l_g \\ &= \frac{B_i}{\mu_i} l_i + \frac{B_g}{\mu_0} l_g \\ &= \frac{l_i}{\mu_i A_i}\phi + \frac{l_g}{\mu_0 A_g}\phi \end{aligned} \tag{4B.3}$$

that is analogous to KVL in electric circuits

This looks like the magnetic analog of KVL, taken around a circuit consisting of a DC source and two resistors. We will therefore exploit this analogy and develop the concept of *reluctance* and *mmf.*





Define reluctance as:

Reluctance defined

$$\mathcal{R} = \frac{l}{\mu A} \tag{4B.4}$$

and magnetomotive force (mmf) as:

Magnetomotive force (mmf) defined

$$\mathcal{F} = Ni \tag{4B.5}$$

then Ampère's Law gives:

Ampère's Law looks like a "magnetic Ohm's Law" for this simple case

$$\begin{aligned} \mathcal{F} &= \left(\mathcal{R}_i + \mathcal{R}_g\right)\phi \\ &= \mathcal{R}\phi \end{aligned} \tag{4B.6}$$

This is analogous to Ohm's law. It should be emphasised that this is only true where $\mu$ is a constant. That is, it only applies when the material is linear or assumed to be linear over a particular region.

The inductance of the toroidal coil is given by the definition of inductance:

The inductance of a toroid using physical characteristics

$$L = \frac{\lambda}{i} = \frac{N\phi}{\sum_l Hl \big/ N} = \frac{N^2\phi}{\mathcal{R}\phi} = \frac{N^2}{\mathcal{R}} \tag{4B.7}$$

Electromechanical devices have one magnetic circuit and at least one electric circuit. The magnetic material serves as a coupling device for power. Such devices include the transformer, generator, motor and meter.

Because magnetic circuits containing ferromagnetic materials are nonlinear, the relationship $\mathcal{F} = \mathcal{R}\phi$ is not valid, since this was derived for the case where $\mu$ is a constant.

Ampère's Law, on the other hand, is always valid, and the concept of magnetic potential will be used where $\mu$ is nonlinear.

The magnetic analog to KVL is Ampère's Law. *What is the magnetic analog to KCL?* In simple systems where the flux path is known, the flux entering a point must also leave it. The analog to KCL for magnetic circuits is therefore Gauss' Law.

Gauss' Law is the magnetic analog of KCL

The two laws we will use for *magnetic circuits* are:

Ampère's Law and Gauss' Law are simple summations for magnetic circuits

$$\sum_l \mathcal{F} = \sum_l U, \quad \text{around a loop } l \tag{4B.8a}$$

$$\sum \phi = 0, \quad \text{at a node} \tag{4B.8b}$$

## Magnetic and Electric Equivalent Circuits

All electromagnetic systems should be reduced to simple magnetic and electric equivalent circuits

To formalise our problem solving capabilities, we will convert every conceivable electromagnetic device into an equivalent magnetic circuit and an equivalent electric circuit. We can analyse such circuits using techniques with which we are familiar. The magnetic circuit for the toroidal coil is:

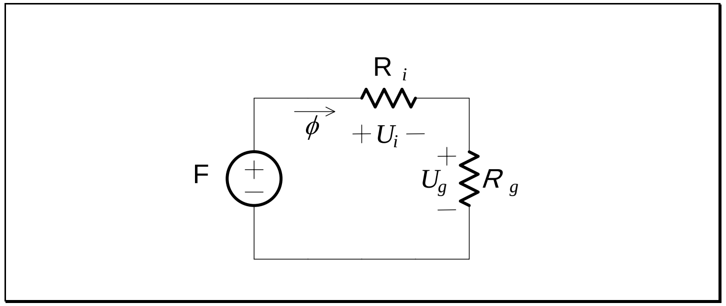



**Figure 4B.2**

There are various ways to analyse the circuit, depending on whether we know the current or flux, but all methods involve Ampère's Law around the loop:

$$\begin{aligned} \mathcal{F} &= U_i + U_g \\ &= H_i l_i + H_g l_g \end{aligned} \tag{4B.9}$$





The electric circuit for the toroidal coil is:

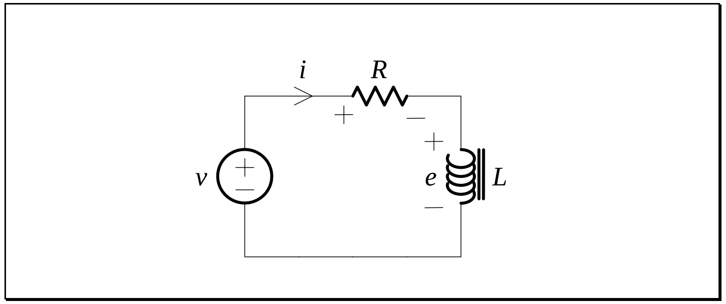



**Figure 4B.3**

KVL around the loop gives:

$$v = Ri + e \tag{4B.10}$$

It is normally a difficult circuit to analyse because of the nonlinear inductance (which must be taken from a $\lambda$-$i$ characteristic).

The voltage source applied to the coil is said to *excite* the coil, and is known as voltage *excitation*. Two special cases of excitation are of particular interest and practical significance – DC excitation and AC excitation.

## DC Excitation

DC excitation refers to the case where a source is applied to the coil which is constant with respect to time.

DC excitation defined

For DC excitation, in the steady-state, the electric circuit is easy to analyse. Faraday's Law for the inductor is:

$$e = \frac{d\lambda}{dt} = \frac{d(Li)}{dt} = L\frac{di}{dt} \tag{4B.11}$$

where $L$ may be nonlinear. Initially, the circuit will undergo a period of *transient* behaviour, where the current will build up and gradually converge to a *steady-state* value. The circuit will be in the steady-state when there is no more change in the current, i.e. $di/dt = 0$. Then Faraday's Law gives the voltage across the inductor as 0 volts (regardless of whether the inductance is linear or not).

KVL around the equivalent circuit then gives:

$$V = RI \tag{4B.12}$$

where we use a capital letter for $I$ to indicate a constant, or DC, current.

Therefore, there is a direct relationship, in the form of Ohm's Law, between the applied voltage and the resultant steady-state current, i.e. the voltage source sets the *current*, so we need to look up the resultant *flux* on the inductor's $\lambda \sim i$ characteristic. This flux is obviously constant with respect to time, since the current is constant with respect to time.

## AC Excitation

AC excitation defined

AC excitation refers to the case where a source is applied to the coil which is continuously changing with respect to time – in most cases the excitation is sinusoidal.

AC excitation analysed without considering the effect of the resistance

For AC excitation, we simplify the analysis by assuming the resistance is negligible. KVL then gives:

$$v = \hat{V}\cos(\omega t) = e = \frac{d\lambda}{dt} \tag{4B.13}$$

and:

$$\lambda = \int_0^t e\,d\tau = \int_0^t \hat{V}\cos(\omega\tau)\,d\tau = \frac{\hat{V}}{\omega}\sin(\omega\tau) \tag{4B.14}$$

Therefore, there is a direct relationship between the applied voltage and the resultant sinusoidal flux, i.e. the voltage source sets the *flux*, so we need to look up the resultant *current* on the inductor's $\lambda \sim i$ characteristic. Even though the





flux is sinusoidal, the resulting current is not sinusoidal, due to the hysteresis characteristic of the ferromagnetic material used to make the inductor. However, the current is periodic, and it does possess *half-wave symmetry*.

## Characteristics

A $B$-$H$ characteristic can be rescaled to give a $\phi$-$U$ characteristic

The $B$-$H$ characteristic can be converted to a $\phi$-$U$ characteristic for a given material:

$$\begin{aligned} \phi &= BA \\ U &= Hl \end{aligned} \tag{4B.15}$$

How a "composite" $B$-$H$ characteristic takes into account the properties of all the materials in a magnetic circuit

For this particular case, there is only one path that the flux takes, so the flux is the same through each material (iron and air). We should, for a given flux, be able to look up on each material's characteristic how much $U$ there is because of this flux. The total $U$ for the magnetic circuit for a given flux is just the addition of the two $U$s. For each value of flux, we can draw the corresponding total value of $U$. The result is a *composite characteristic*. It is useful if there is more than one ferromagnetic material in the circuit.

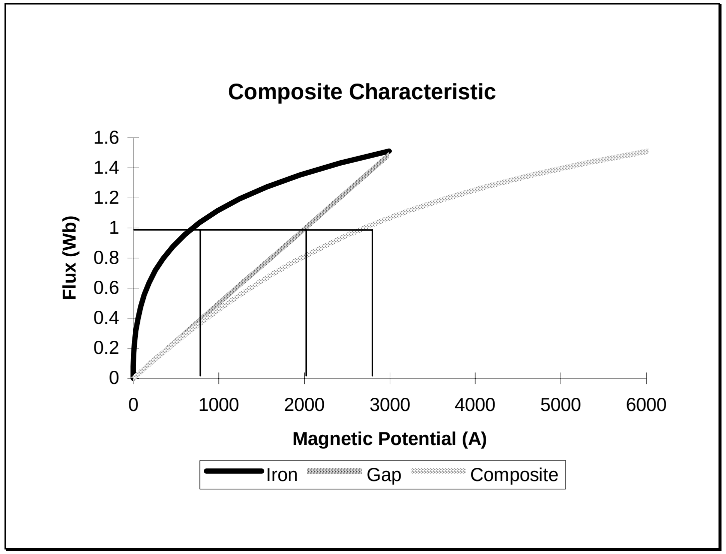



**Figure 4B.4**

## Determining $\mathcal{F}$ given $\phi$

If we are given the flux, then the potentials can be obtained from a $\phi$-$U$ characteristic.

If given $\phi$, then look up $U$

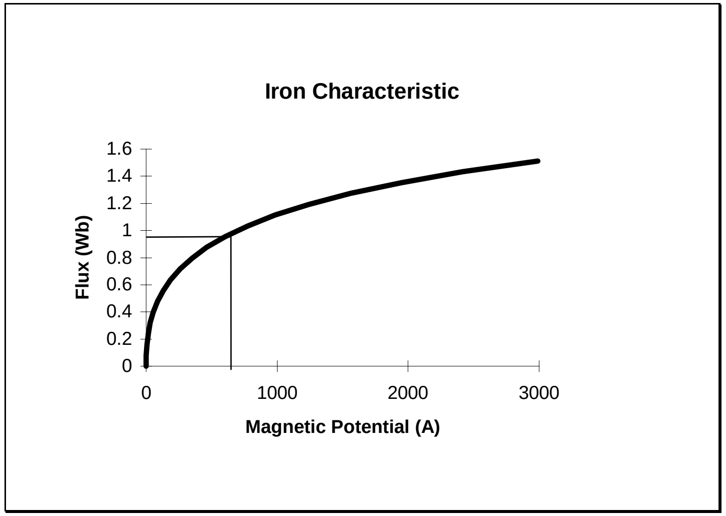



**Figure 4B.5**

For air gaps, we don't need a characteristic, since it is linear. We use:

For air gaps, the characteristic is a straight line, so use the equation

$$U_g = \mathcal{R}_g \phi = \frac{l_g}{\mu_0 A_g}\phi \tag{4B.16}$$

Ampère's Law is applied, and we get:

$$\mathcal{F} = U_i + U_g = Ni \tag{4B.17}$$

Given $\mathcal{F}$, the choice of $N$ and $i$ is dictated by other considerations

The number of turns and current can be chosen to suit the physical conditions, e.g. small wire (low current rating) with lots of turns or large wire (high current rating) with a few turns.





## Determining $\phi$ given $\mathcal{F}$ (Load Line)

Given $\phi$, the “load line” determines $\mathcal{F}$

An iterative procedure may be carried out in this case. A better way is to use a concept called the load line. (A gap is said to “load” a magnetic circuit, since it has a high $U$). This concept is used in graphical analysis of nonlinear systems. The load line, being linear, must be derived from a linear part of the system. In a magnetic circuit, the air gap has a linear relationship between $\phi$ and $U$.

Using Ampère's Law, we get:

The load line equation for a magnetic circuit with one gap

$$\mathcal{F} = U_i + U_g = U_i + \mathcal{R}_g \phi$$
$$\phi = -\frac{1}{\mathcal{R}_g}\left(U_i - \mathcal{F}\right) \tag{4B.18}$$

This is the equation of the load line. The unknown quantities are $\phi$ and $U_i$. This equation must be satisfied at all times (Ampère's Law is always obeyed). There are two unknowns and one equation. *How do we solve it?*

We need another equation. The other equation that must be obeyed at all times is one which is given in the form of a graph – the material’s characteristic. It is nonlinear.

The load line and the ferromagnetic material's characteristic are both satisfied at the point of intersection

To solve the system of two equations in two unknowns, we plot the load line on the characteristic. Both graphs are satisfied at the point of intersection. We can read off the flux and potential.

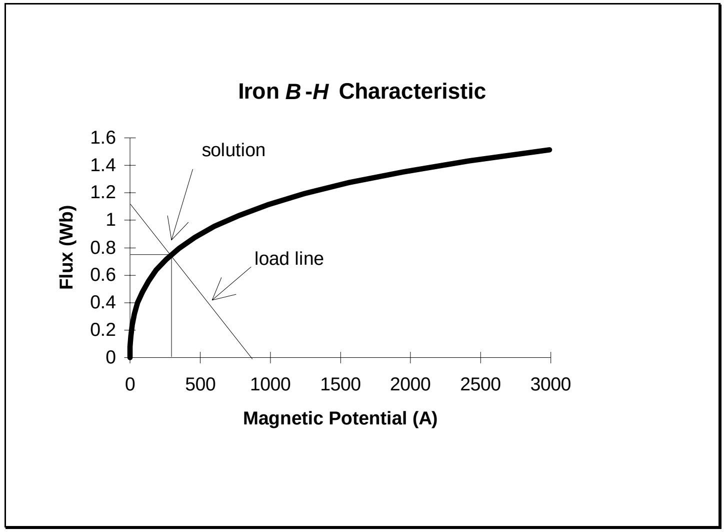



**Figure 4B.6**

If the material is specified in terms of a *B-H* characteristic, then the equation of the line becomes:

$$\mathcal{F} = H_i l_i + \frac{B_g}{\mu_0} l_g$$

$$B_g = -\frac{\mu_0 l_i}{l_g}\left(H_i - \frac{\mathcal{F}}{l_i}\right) \tag{4B.19}$$





## Equivalent Circuit of a Permanent Magnet

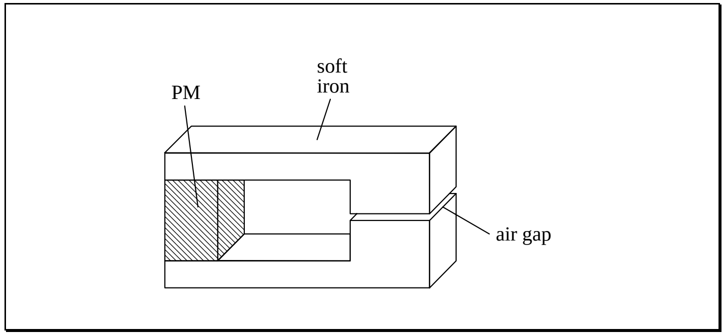



**Figure 4B.7**

A permanent magnet (PM) produces flux without mmf

In this case, we know the mmf – it is zero since there is no applied current. The method of finding the flux in a magnetic circuit containing a permanent magnet (PM) therefore follows the same procedure as above. We ignore the soft iron (it has infinite permeability compared to the air gap):

Ampère's Law (load line equation) for a PM

$$0 = U_m + U_g = U_m + \mathcal{R}_g \phi$$

$$\phi = -\frac{1}{\mathcal{R}_g} U_m \tag{4B.20}$$

For a PM, only one part of the hysteresis loop is valid

To solve for the flux, we need the PM's characteristic. We can see that for a positive flux, the magnet exhibits a negative potential. This makes sense because we have always assumed that the magnetic potential is a drop. A negative drop is equivalent to a rise – a PM is a source of potential and therefore flux. The load line intersects the *B-H* hysteresis loop (not the normal magnetization characteristic) to give the operating point (or *quiescent* point, or *Q*-point for short).

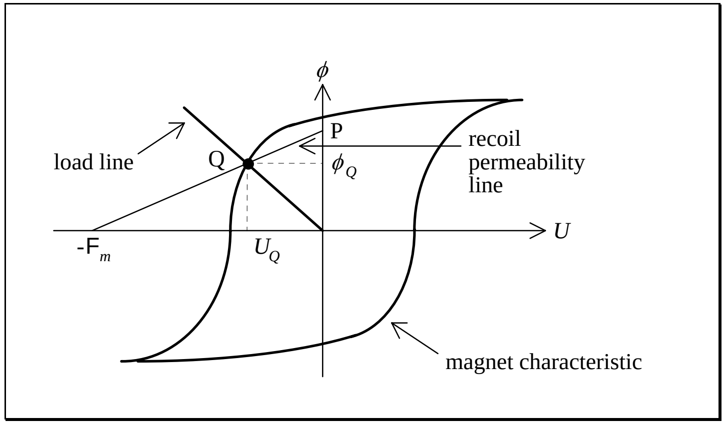



**Figure 4B.8**

The operating point of a PM moves along the “recoil line” for changes in “load”

A PM exhibits hysteresis, so when the gap is replaced with a soft iron keeper, the characteristic is not traced back. The operating point moves along another line called the recoil permeability line (PQ in Figure 4B.8) to P. Subsequent opening and closing of the gap will cause the operating point to move along PQ. A good permanent magnet will operate along PQ almost continuously.

If the operating point always lies between P and Q, then we can use the equation for this straight line in the analysis. This is also equivalent to modelling the PM with linear elements.

A linear model of a PM hysteresis loop

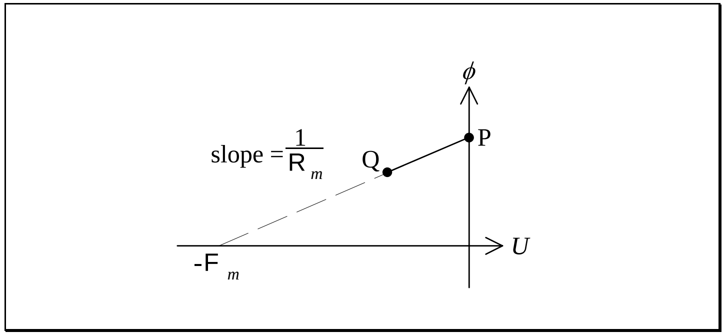



**Figure 4B.9**





The PM linear model is therefore:

A linear circuit model of a PM

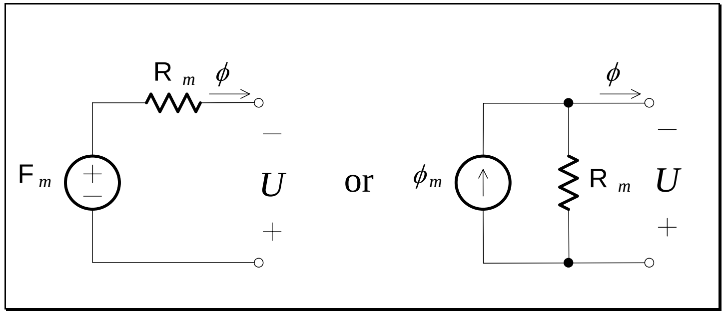



**Figure 4B.10**

The linear circuit model for the PM is only valid for load lines that cross the recoil line between P and Q.

**Example – Determine $\mathcal{F}$ given $\phi$**

Consider the following electromagnetic system:

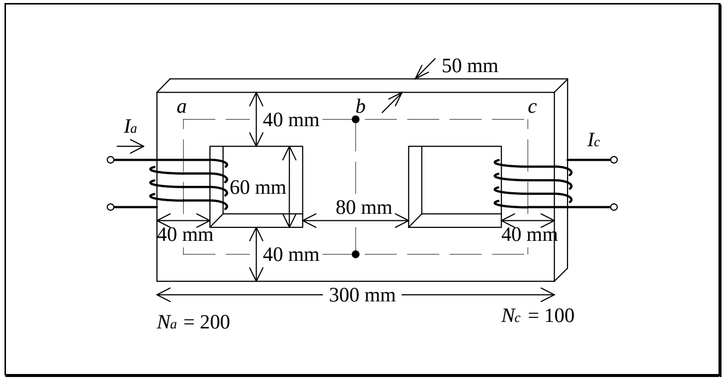



**Figure 4B.11**

Given:

- The core is laminated sheet steel with a stacking factor = 0.9.
- $\phi_a = 1.8$ mWb, $\phi_b = 0.8$ mWb, $\phi_c = 1$ mWb.
- $I_a$ is in the direction shown.





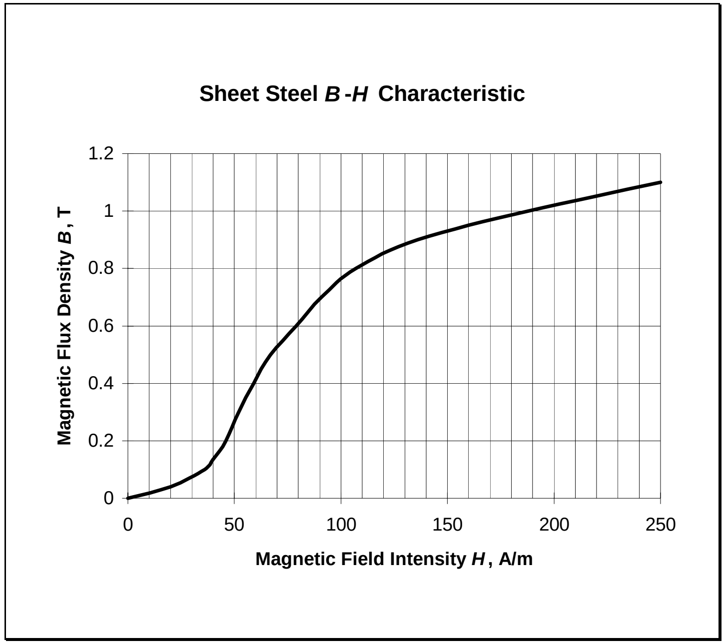



**Figure 4B.12**

Draw the magnetic equivalent circuit.

Show the directions of $\phi_a$, $\phi_b$ and $\phi_c$.

Determine the magnitude of $I_a$ and the magnitude and direction of $I_c$.

*Solution:*

The magnetic equivalent circuit is:

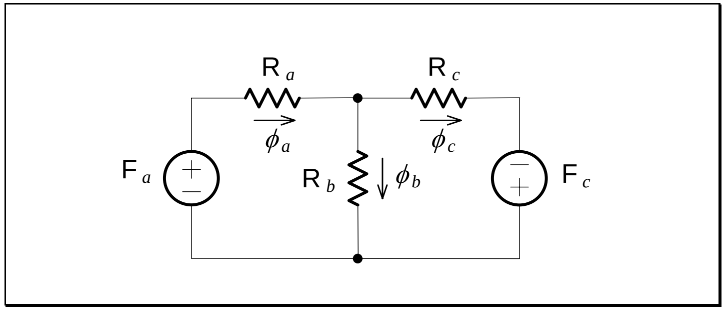



**Figure 4B.13**

As the cross sectional area is uniform, branches *a* and *c* are taken right up to the middle of the centre limb. Therefore: $l_a = 0.36$ m, $l_b = 0.1$ m and $l_c = 0.36\,\text{m}$.

From the *B*-*H* characteristic, since $B = \phi/A$ and $\phi$ is given, we just look up:

$$H_a = 200\ \text{Am}^{-1},\ H_b = 50\ \text{Am}^{-1}\ \text{and}\ H_c = 75\ \text{Am}^{-1}.$$

Applying Ampère's Law around the left hand side (LHS) loop gives:

$$\mathsf{F}_a = U_a + U_b = H_a l_a + H_b l_b = 200\times 0.36 + 50\times 0.1 = 77\ \text{A}$$

Applying Ampère's Law around the right hand side (RHS) loop gives:

$$\mathsf{F}_c = U_c - U_b = 75\times 0.36 - 50\times 0.1 = 22\ \text{A}$$

Therefore: $I_a = \mathsf{F}_a/N_a = 0.385\ \text{A}$ and $I_c = \mathsf{F}_c/N_c = 0.22\ \text{A}\,(\rightarrow)$.





**Example – Permanent Magnet Operating Point**

Consider the following permanent magnet (PM) arrangement:

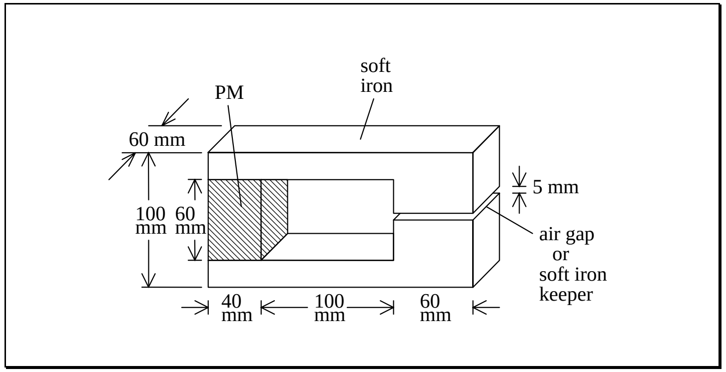



**Figure 4B.14**

The PM has the following *B*-*H* characteristic:

PMs "operate" with a negative **H**

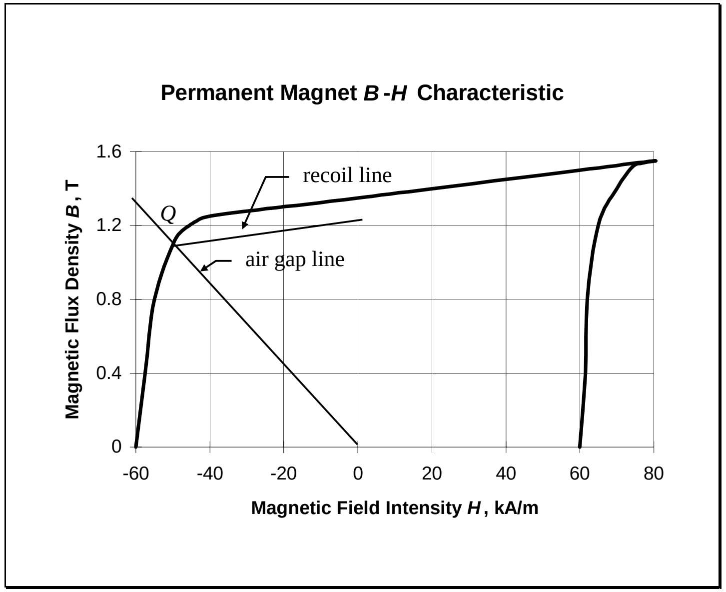



**Figure 4B.15**

a) Remove keeper. Determine gap flux density $B_g$.

b) Insert keeper. Determine residual flux density.

Ignore leakage and fringing flux.

Assume $\mu_{recoil} = 2\mu_0$ and $\mu_{soft\ iron} = \infty$.

$m \equiv$ magnet, $i \equiv$ soft iron, $g \equiv$ gap.

*Solution:*

a) As there is no externally applied current, Ampère's Law gives:

$$U_m + \cancel{U_i} + U_g = 0 \quad (U_i = 0 \text{ (since } \mu_i = \infty))$$

Therefore:

$$U_m = -U_g = -\mathcal{R}_g \phi \text{ or}$$

The "air gap" line for a PM

$$\phi = -\frac{1}{\mathcal{R}_g} U_m \tag{4B.21}$$

i.e. the equation of a straight line (the *air gap line* or *load line*).

If the *B*-*H* characteristic of the PM is re-scaled to give a $\phi_m$-$U_m$ characteristic, the load line (slope = $-1/\mathcal{R}_g$) intersects it at the "operating point" $Q$. Otherwise, as $\phi = BA$ and $U = Hl$:

rewritten in a form suitable for plotting on a *B*-*H* characteristic

$$B_m = -\mu_0 \frac{A_g l_m}{A_m l_g} H_m \tag{4B.22}$$

From the dimensions given (and $\mu_0 = 4\pi \times 10^{-7}$ Hm$^{-1}$) we get $B_m = -72\pi \times 10^{-7} H_m$. Therefore, at $H_m = -60 \times 10^3$ Am$^{-1}$, $B_m = 1.355$ T. We draw the load line through this point and the origin. At the operating point $Q$, $B_m = 1.12$ T. Therefore $B_g = \frac{A_m}{A_g} B_m = \frac{2}{3} B_m = 0.745$ T.





b) When the keeper is reinserted:

We draw a line from the $Q$-point, with slope $2\mu_0$ to the $B$ axis. The intersection gives the residual flux density $B_m$.

*or*:

The change in magnetic field intensity is $\delta H_m = 50 \times 10^3$. The flux density change is $\delta B_m = 2\mu_0 \delta H_m = 0.13$ T. Therefore residual $B_m = 1.12 + 0.13 = 1.25$ T.

**Example – Permanent Magnet Minimum Volume**

Consider the PM and characteristic of the previous example.

The flux in the PM is:

$$B_g A_g = B_m A_m$$

and the potential across the PM is:

$$H_g l_g = -H_m l_m$$

The volume of the PM can therefore be expressed as:

The volume of a PM can be minimised by careful choice of the *Q*-point

$$V_m = A_m l_m = l_g A_g \frac{H_g B_g}{|H_m B_m|} \tag{4B.23}$$

which is a minimum if $|H_m B_m|$ is a maximum.

If we plot $B$-$|HB|$ for the magnet we get:

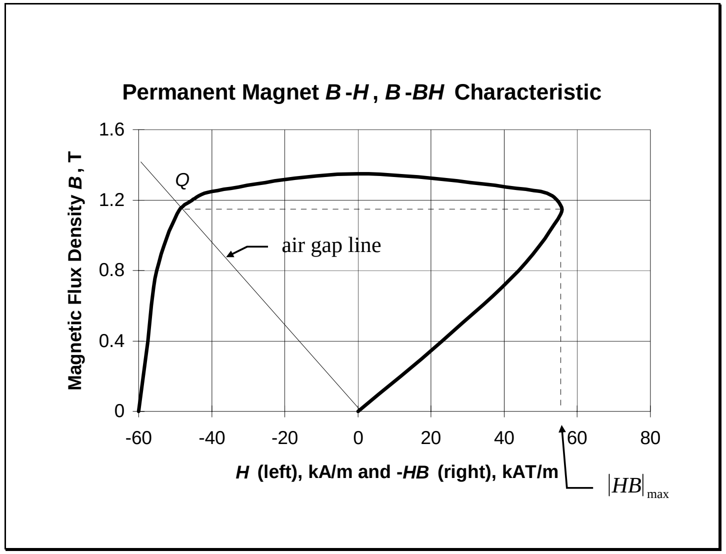



**Figure 4B.16**

Figure 4B.16 shows how the plot is constructed from the PMs *B-H* characteristic. The plot shows that if we choose a PM with minimum volume (small cost – PMs are expensive) then we should choose an operating point given by *Q*. The previous example was therefore a good design.





## Summary

- We define the reluctance of a magnetic material as: $\mathcal{R} = \dfrac{l}{\mu A}$.
- Wed define the magnetomotive force (mmf) of a coil as: $\mathcal{F} = Ni$.
- For uniform magnetic fields and linear magnetic material, Ampère's Law is the magnetic analog of Ohm's Law: $\mathcal{F} = \mathcal{R}\phi$.
- The inductance of a toroidal coil is given by: $L = \dfrac{N^2}{\mathcal{R}}$.
- Ampère's Law and Gauss' Law are the magnetic analogs of Kirchhoff's Voltage Law and Kirchhoff's Current Law, respectively: $\sum_l \mathcal{F} = \sum_l U$ and $\sum \phi = 0$.
- We can convert every electromagnetic device into an equivalent magnetic circuit and an equivalent electric circuit.
- For ferromagnetic materials, we use the nonlinear *B-H* characteristic analyse the magnetic circuit.
- A load line represents a linear relationship between circuit quantities and is usually graphed on a circuit element's characteristic to determine the operating point, or *Q*-point.
- A permanent magnet exhibits an internal negative magnetic potential and can be used to create flux in a magnetic circuit without the need for an external mmf. The operating point of a magnetic circuit that uses a permanent magnet can be optimised to minimise the volume of permanent magnetic material.

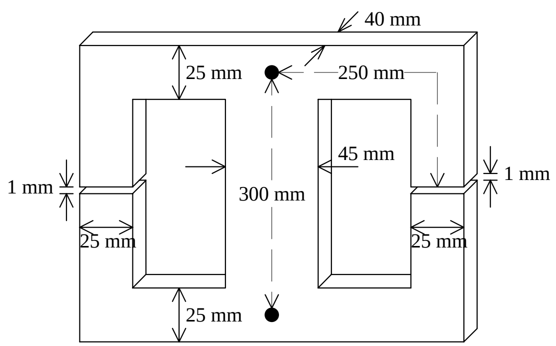
## References

Plonus, Martin A.: *Applied Electromagnetics*, McGraw Hill Kogakusha, Ltd., Singapore, 1978.

## Problems

**1.**

Consider the following magnetic structure:




The normal magnetisation characteristic of the core material is,

| $H\left(\text{Am}^{-1}\right)$ | 100 | 200 | 280 | 400 | 600 | 1000 | 1500 | 2500 |
|---|---|---|---|---|---|---|---|---|
| $B\left(\text{T}\right)$ | 0.51 | 0.98 | 1.20 | 1.37 | 1.51 | 1.65 | 1.73 | 1.78 |

Determine the flux density in the centre limb and the necessary mmf for a winding on the centre limb:

(a) For a flux density of 1.2 T in each air gap,

(b) For a flux density of 1.2 T in one air gap when the other is closed with a magnetic material of the same permeability as the core material.





**2.**

Consider the magnetic structure of Q1. The centre limb has a winding of 500 turns, carrying 1 A.

Draw the equivalent magnetic circuit and determine the total reluctance of the circuit and the flux density in the RHS air gap for:

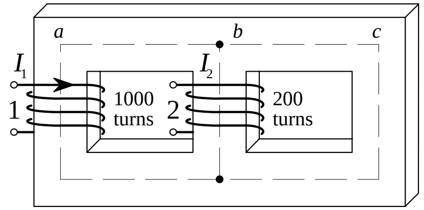
(a) Both air gaps open,

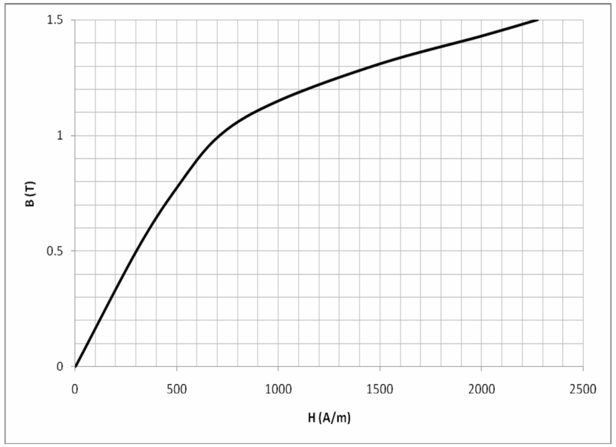
(b) LHS air gap closed.

Assume a constant permeability $\mu = 5\times10^{-3}\ \text{Hm}^{-1}$.

**3.**

Consider the magnetic core and magnetization characteristic shown.




The core has a uniform cross sectional area (csa) $A = 6.4 \times 10^{-3}\ \text{m}^2$, $l_a = l_c = 0.88$ m and $l_b = 0.16$ m. Coil 1 has $I_1 = 0.5$ A (DC).

(a) Determine the magnitude and direction of $I_2$ needed to give $\phi_b = 0$.

(b) Develop expressions for $L_{21}$ and $L_{12}$, and calculate their value for the currents and fluxes determined in (a).





**4.**

Refer to the example *Permanent Magnet Operating Point*.

The air gap flux was assumed to be confined within the air gap. Consider now the leakage flux between the upper and lower horizontal sections of the soft magnetic material, and assume that the leakage flux density is uniform.

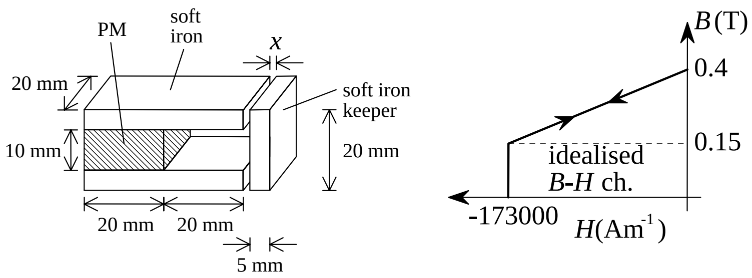
(a) Determine the flux density in the PM when the keeper is removed. Compare the air gap flux density with that calculated in the example.

(b) The leakage flux may be reduced by placing the PM closer to the air gap. Sketch an improved arrangement of the system.

# 4B.26

**5.**

The PM assembly shown is to be used as a door holder (keeper attached to the door, remainder attached to the frame).




Assume that for soft iron $\mu = \infty$.

(a) Derive a linearised magnetic equivalent circuit (neglect leakage and fringing), and determine the maximum air gap length $x_{\max}$ for which it is valid.

(b) The linear model used in (a) assumes that the magnet will be demagnetised when $x > x_{\max}$. Show that the leakage reluctance between the upper and lower soft iron pieces is low enough to prevent demagnetisation from occurring.

# Lecture 5A – Graphical Analysis

*The static characteristic. The dynamic characteristic. The transfer characteristic. Graphical analysis. The small signal diode model. The large signal diode model.*

## The Static Characteristic

Any linear resistive circuit can be reduced to an equivalent circuit containing one source and one resistor. When the source is a voltage, the reduction is obtained using Thévenin's theorem.

Use Thévenin's Theorem to simplify the *linear* parts of a circuit

Consider the following circuit:

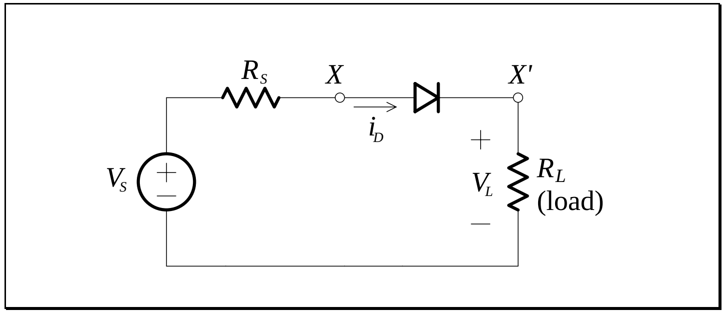



**Figure 5A.1**

The equivalent circuit (as far as the diode is concerned) can be found using Thévenin's theorem. *(Look into the circuit from the diode terminals. What do you see?)* The Thévenin equivalent circuit for the diode is:

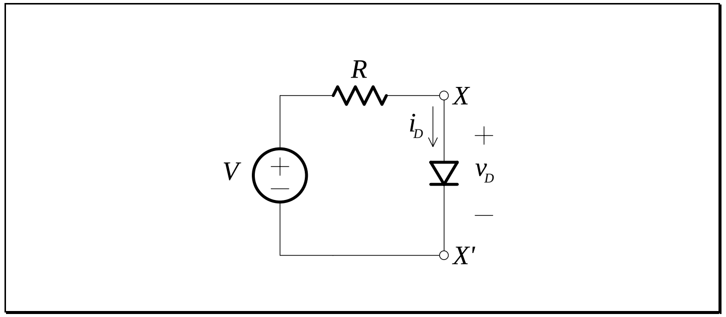



**Figure 5A.2**

*Verify that the Thévenin voltage and Thévenin resistance in this case are given by:*

$$V = V_s \tag{5A.1a}$$

$$R = R_S + R_L \tag{5A.1b}$$

KVL around the loop gives:

$$v_D = V - Ri_D \tag{5A.2}$$

which, when rearranged to make $i_D$ the subject, gives:

$$i_D = -\frac{1}{R}\left(v_D - V\right) \tag{5A.3}$$

The "load line" is derived using linear circuit theory

When graphed, we call it the load line. It was derived from KVL, and so it is always valid. *(Compare with the load line in magnetics which was obtained from Ampère's Law).*

The load line gives a relationship between $i_D$ and $v_D$ that is determined purely by the external circuit. The diode's characteristic gives a relationship between $i_D$ and $v_D$ that is determined purely by the geometry and physics of the diode.





The "load line" and device characteristic intersect at the $Q$ point

Since both the load line and the characteristic are to be satisfied, the only place this is possible is the point at which they meet. This point is called the quiescent point, or $Q$ point for short. It is only valid for DC conditions (*Why?*):

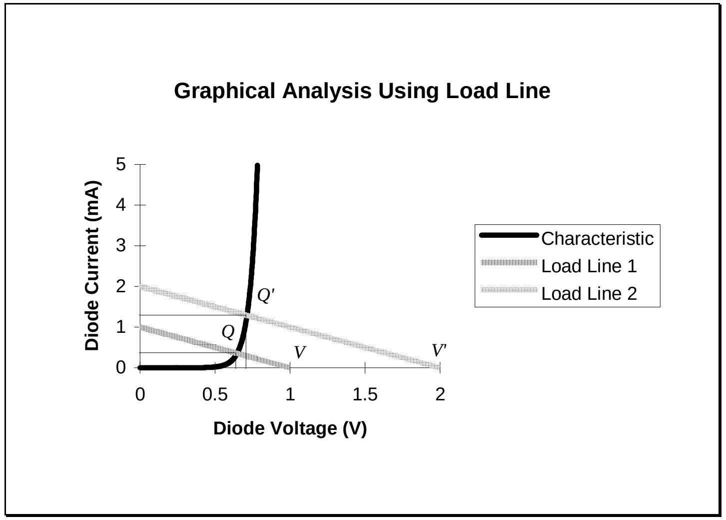



**Figure 5A.3**

If the source voltage is increased the Thévenin voltage changes to $V'$ and the operating point to $Q'$ (the DC load line is shifted up).

## The Dynamic Characteristic

The dynamic characteristic gives directly the relationship between the diode current and the Thévenin voltage ($I_D \sim V$):

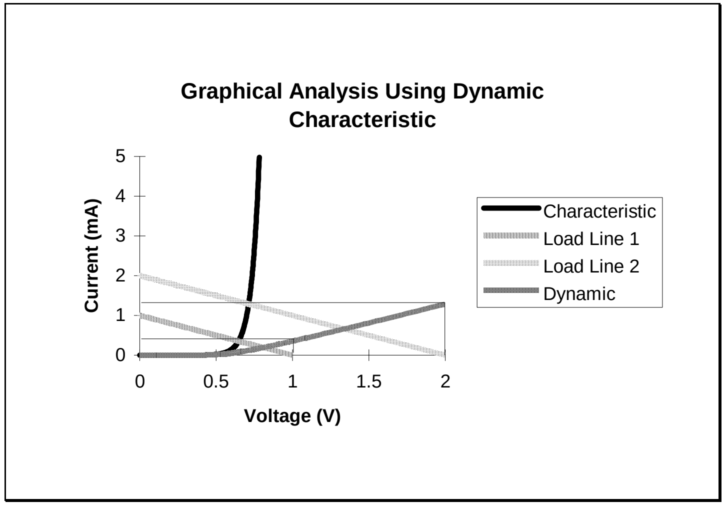



**Figure 5A.4**

## The Transfer Characteristic

A transfer characteristic ($V_L \sim V$) is obtained by graphing the output voltage versus the input voltage (it shows how the input is transferred to the output). It is obtained directly from the dynamic characteristic since $V_L = R_L i_D$.





## Graphical Analysis

To analyse the effect of applying a voltage that contains a DC and AC component, we generally have to use a graphical technique:

We can use graphical analysis on non-linear systems where time is a variable

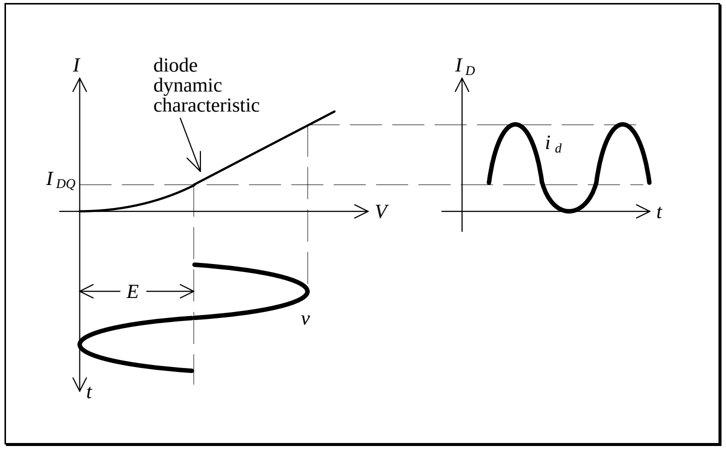



**Figure 5A.5**

From this, we can determine the total current in the circuit, as well as the voltage across the diode:

$$v_D = r_{fd} i_d + V_{DQ} \tag{5A.4}$$

Note that $V_{DQ} \neq E$ in Figure 5A.5.

## The Small Signal Diode Model

With "small signals", we can linearise a non-linear element and then use superposition

If the voltage contains an alternating component that is very small relative to the DC voltage, then the circuit can be analysed using the principle of superposition. The AC component is termed "a small signal". Superposition can only be performed when the system is linear. For the small signal, a linear diode model is used. For a source of:

$$v_s = E_s + \hat{v}_s \sin \omega t \tag{5A.5}$$

the Thévenin equivalent circuit can be split into two separate circuits:

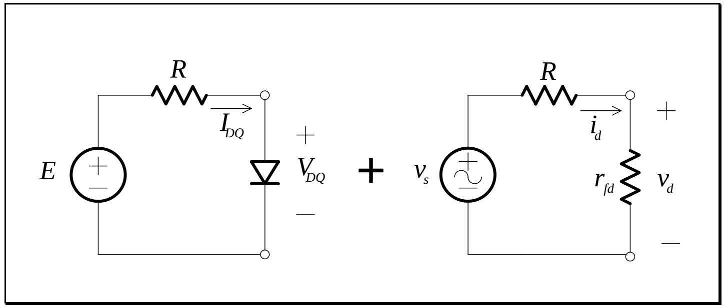



**Figure 5A.6**

The AC equivalent circuit contains small signal parameters of the diode equivalent circuit. The resistance is the inverse of the slope at the *Q* point, the voltage is the voltage axis intercept of the tangent at the *Q* point.

For the model to be valid, the AC signal must be small, so that the tangent to the curve approximates the curve. This is the small signal approximation. You can see the effects of having a large signal using the dynamic or transfer characteristic.

We determine the voltage across the load and the current through it by:

$$i_L = I_{DQ} + i_d \tag{5A.6a}$$

$$v_L = R_L i_L \tag{5A.6b}$$





## The Large Signal Diode Model

With an alternating source only, we can analyse effects such as cut-in angle using a large signal equivalent circuit of the diode:

A suitable linear model of the diode can be used for AC sources

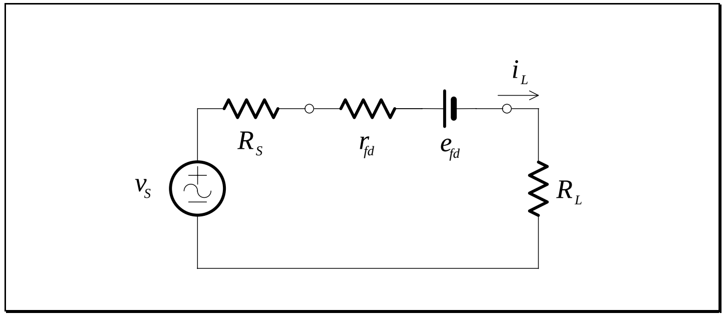



**Figure 5A.7**

Our model is valid only when the diode is conducting, so we will examine the circuit when the diode just reaches the threshold of conduction (but is still zero).

At this point:

$$i_L = \frac{\hat{v}_S \sin \omega t - e_{fd}}{R_S + R_L + r_{fd}} = 0 \tag{5A.7}$$

which means that:

$$\hat{v}_S \sin \omega t = e_{fd}$$

$$\omega t = \gamma = \sin^{-1}\left(\frac{e_{fd}}{\hat{v}_S}\right) \tag{5A.8}$$

For example, if $\hat{v}_S = 5\text{ V}$ and $e_{fd} = 0.6\text{ V}$, then $\gamma = \sin^{-1}(0.6/5) = 6.9°$. This is a small angle, but it is still noticeable on a DSO. The angle gets bigger for a smaller amplitude voltage source.

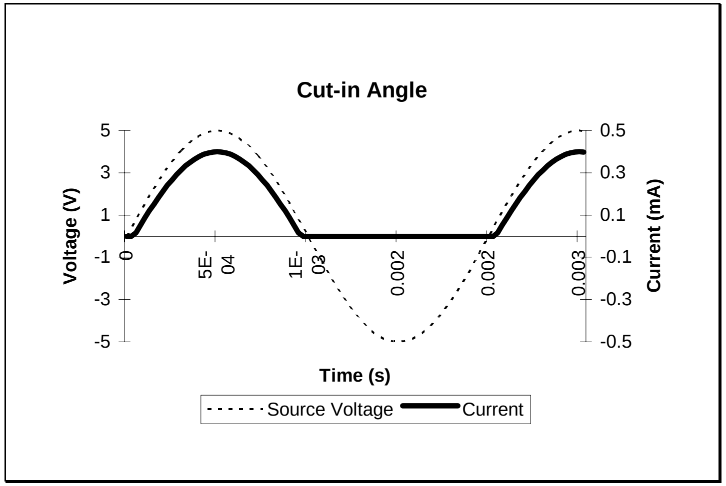



**Figure 5A.8**





## Summary

- We can use Thévenin's theorem to simplify the linear parts of a circuit. Analysis of the linear circuit leads to the load line, which can then be graphed on a nonlinear circuit element's characteristic to obtain the operating point (also called the *Q*-point).
- We can create dynamic characteristics and transfer characteristics of circuits with nonlinear elements using graphical techniques.
- We can conduct separate DC and AC analyses of a circuit if the circuit is linear, using the principle of superposition. For nonlinear circuit elements, we first find the DC operating point, then linearise the nonlinear element's characteristic to obtain a "small signal" model. We can then conduct AC circuit analysis. The analysis will only be valid for small deviations about the DC operating point.

## References

Sedra, A. and Smith, K.: *Microelectronic Circuits*, Saunders College Publishing, New York, 1991.

# Lecture 5B – Field Energy

*Energy stored in the magnetic field. Electric field energy. Total field energy. Hysteresis losses. Eddy currents.*

## Energy Stored in the Magnetic Field

The toroid (equivalent to an infinitely long solenoid) is ideal for determining the magnetising characteristic of specimens (no end effects).

A toroid exhibits no end effects – like an infinitely long solenoid

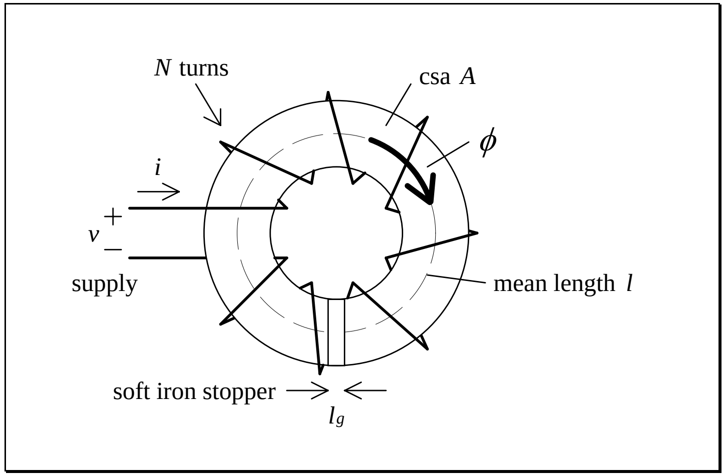



**Figure 5B.1**

The direction of the field in the iron is given by the right hand screw rule.

KVL gives:

$$\begin{aligned} v &= Ri + e \\ &= Ri + \frac{d\lambda}{dt} \end{aligned} \tag{5B.1}$$

Therefore, the electric power delivered by the source is:

The power delivered to a solenoid

$$p = vi = Ri^2 + i\frac{d\lambda}{dt} \quad \text{W} \tag{5B.2}$$

Some of the power delivered to a solenoid is dissipated as heat

Positive values for the terms in Eq. (5B.2) represent power delivered *by* the source. The first term, involving the resistance of the winding, $R$, is always positive. *(Why?)* This term therefore represents a power dissipation, or loss, in the form of heat that always exists – regardless of the current direction. In words, Eq. (5B.2) reads as:

*input power = losses* (5B.3)
*+ power associated with rate of change of flux*

Since there is no electrical or mechanical output, this equation is really just a statement of the conservation of energy: *energy in = energy out*.

Some of the energy delivered to a solenoid is stored in the magnetic field

A negative number in Eq. (5B.2) represents power delivered *to* the source. The power associated with the rate of change of flux is therefore not a loss, since it can have a negative value. It represents power either stored or delivered by the field.

Consider first a DC supply, with the current positive. When it is switched on, the magnetic field must increase from 0 to some value. This is a positive rate of change of flux linkage, so the last term in Eq. (5B.2) is positive – power has been delivered by the source to establish the field.

For steady state DC, the energy stored in the field is constant - the power to the field is zero

In the steady state, the current will be a constant, so the flux linkage will not change – no more power is delivered to the field and the only loss is resistance.

Stored magnetic field energy is released when the current (which causes the field) is interrupted

When switching the supply off, the field must return to zero from a positive value. This is a negative rate of change of flux linkage, so the last term is negative – power has been delivered to the source from the field. The stored field energy is returned to the system in some form. (Note what happens when you switch off an inductive load – the energy stored in the field is dissipated as an arc in the switch).





Consider a sinusoidal supply. The power delivered by the source will vary with time. Firstly, consider the instantaneous power in the resistor:

The instantaneous power delivered to a resistor from a sinusoidal supply

$$\begin{aligned} p_R &= Ri^2 \\ &= R\hat{I}^2\cos^2\omega t \\ &= R\hat{I}^2\frac{1}{2}\left(1+\cos 2\omega t\right) \end{aligned} \tag{5B.4}$$

The average power dissipated in the resistor is:

The average power delivered to a resistor from a sinusoidal supply

$$\begin{aligned} P_R &= \frac{1}{T}\int_0^T \frac{R\hat{I}^2}{2}\left(1+\cos 2\omega t\right)dt \\ &= \frac{R\hat{I}^2}{2T}\left[t+\frac{1}{2\omega}\sin 2\omega t\right]_0^T \\ &= \frac{R\hat{I}^2}{2} = RI_{RMS}^2 \end{aligned} \tag{5B.5}$$

This term is always positive, and should be familiar. Assuming a linear inductance, the instantaneous power delivered *to* the field is:

The instantaneous power delivered to an inductor from a sinusoidal supply

$$\begin{aligned} p_L &= i\frac{d\lambda}{dt} \\ &= \hat{I}\cos\omega t\frac{d}{dt}\left(L\hat{I}\cos\omega t\right) \\ &= -\omega L\hat{I}^2\cos\omega t\sin\omega t \\ &= -\omega L\hat{I}^2\frac{1}{2}\sin 2\omega t \end{aligned} \tag{5B.6}$$

We can now derive the average power delivered to the inductor:

The average power delivered to an inductor from a sinusoidal supply

$$\begin{aligned} P_L &= \frac{1}{T}\int_0^T \frac{-\omega L\hat{I}^2}{2}\sin 2\omega t dt \\ &= \frac{-\omega L\hat{I}^2}{2T}\left[\frac{-1}{2\omega}\cos 2\omega t\right]_0^T \\ &= 0 \end{aligned} \tag{5B.7}$$

The *average* power is zero. The *instantaneous* power tells us that power constantly goes to and from the field in a sinusoidal fashion. No power is lost in the field because the *average* power delivered is zero.

Therefore, we can say that the magnetic field, at any instant, is storing energy away at a rate given by:

Instantaneous field power defined

$$p_{field} = i\frac{d\lambda}{dt} \tag{5B.8}$$

The energy stored in the magnetic field, in a given time interval, is:

Field energy defined

$$W_f = \int_{t_1}^{t_2} p_{field} dt = \int_{t_1}^{t_2} i\frac{d\lambda}{dt}dt = \int_{\lambda_1}^{\lambda_2} i d\lambda \tag{5B.9}$$

Applying Gauss' law and Ampère's law around the magnetic circuit, we get:

$$\begin{aligned} i &= \frac{Hl}{N} \\ d\lambda &= N d\phi = NAdB \end{aligned} \tag{5B.10}$$





which means the field energy can be expressed by:

$$\begin{aligned} W_f &= \int_{\lambda_1}^{\lambda_2} i d\lambda \\ &= \int_{B_1}^{B_2} \frac{Hl}{N} NAdB = lA\int_{B_1}^{B_2} HdB \end{aligned} \tag{5B.11}$$

The energy per unit volume is therefore:

Field energy density defined

$$\frac{W_f}{V} = \int_{B_1}^{B_2} HdB \quad \text{Jm}^{-3} \tag{5B.12}$$

Integrating the energy by parts (since current is a function of flux linkage – the magnetisation characteristic), we obtain:

$$\begin{aligned} W_f &= \int_{\lambda_1}^{\lambda_2} i d\lambda \\ &= \left[i\lambda\right]_{\lambda_1}^{\lambda_2} - \int_{i_1}^{i_2} \lambda di \end{aligned} \tag{5B.13}$$

The last term in Eq. (5B.13) has the units of energy, but is not directly related to the energy stored by the magnetic field. We define a new quantity called the magnetic field *co-energy* to be:

Field co-energy defined

$$W_f' = \int_{i_1}^{i_2} \lambda di \tag{5B.14}$$

With this definition, and using Eq. (5B.13), we get the relationship:

$$\begin{aligned} \text{energy} + \text{co-energy} &= i_2\lambda_2 - i_1\lambda_1 \\ W_f + W_f' &= i_2\lambda_2 - i_1\lambda_1 \end{aligned} \tag{5B.15}$$

A graphical interpretation of this relationship is shown below:

Energy and co-energy shown graphically

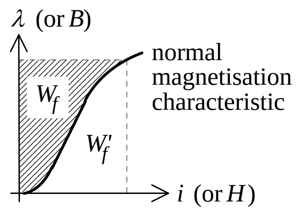



**Figure 5B.2**

To determine the total energy stored in a magnetic field, that was brought from zero to a steady value, we set the flux and current to zero at time $t_1$:

The total energy and co-energy stored in a magnetic system

$$W_f = \int_0^\lambda i\,d\lambda$$
$$W'_f = \int_0^I \lambda\,di$$
$$W_f + W'_f = I\lambda \tag{5B.16}$$

For linear systems, which have straight line $\lambda - i$ characteristics, the energy and co-energy are equal (they are triangles) and are given by:

The energy and co-energy are equal for linear systems – and are easily calculated

$$W_f = W'_f = \frac{1}{2}I\lambda = \frac{1}{2}LI^2 \quad (\text{since } \lambda = LI) \tag{5B.17}$$

If the soft iron keeper is removed from the toroid to create an air gap, then two different uniform fields are created – one in the iron and one in the gap. Each will have its own energy. Assume that the permeability of the iron is a constant (this corresponds to linearising the *B-H* characteristic).





The energy density of a field with constant permeability is:

$$\frac{W_f}{V} = \int_0^B H\,dB = \int_0^B \frac{B}{\mu}\,dB = \frac{B^2}{2\mu} \ \text{Jm}^{-3} \tag{5B.18}$$

The energy density of a linear system

The energies in the gap and iron are therefore:

$$\frac{W_{firon}}{V_{iron}} = \frac{B^2}{2\mu_0\mu_r}$$
$$\frac{W_{fgap}}{V_{gap}} = \frac{B^2}{2\mu_0} = \mu_r \frac{W_{firon}}{V_{iron}} \tag{5B.19}$$

The energy density of an air gap is much larger than the energy density of iron

For practical values of permeability and volume, the energy of the field in the air gap is much larger than that in the iron. This is usually what we want, since it is in the gap where the magnetic field is used (e.g. motor, generator, meter).

## Electric Field Energy

The same analysis as above can be performed with a capacitor to calculate the energy per unit volume stored in the electric field:

$$\frac{W_f}{V} = \int_0^D E\,dD \ \text{Jm}^{-3} \tag{5B.20}$$

The energy density for electric fields

## Total Field Energy

In general, the total energy per unit volume stored in a system is:

$$\frac{W_f}{V} = \int_0^D E\,dD + \int_0^B H\,dB \ \text{Jm}^{-3} \tag{5B.21}$$

The energy density considering all fields

### Hysteresis Losses

With ferromagnetic cores, when the applied field is varied, energy is dissipated as heat during the realignment of the domain walls and a power loss results.

We know that the field energy density is the area under the *B-H* loop. Let's see what happens to the field energy when the operating point goes around the *B-H* loop.

Imagine the hysteresis loop has already been established, and we are at point *a*.

The area of the *B-H* hysteresis loop represents energy density *loss* in a magnetic system

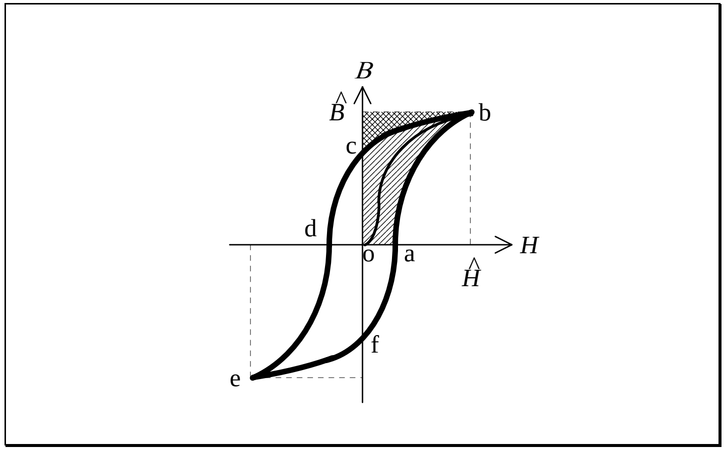



**Figure 5B.3**

At this point, the instantaneous energy supplied to the field is zero. If we increase the applied field strength $H$ to the point $b$, then the energy supplied to the field is:

$$W_{ab} = V\int_0^{\hat{B}} H dB \tag{5B.22}$$





When the applied field strength is brought back to zero at point $c$, energy is returned to the source, but there has been some energy loss.

$$W_{bc} = V\int_{\hat{B}}^{B_c} H dB \tag{5B.23}$$

The loss is the diagonally striped part of the hysteresis loop. In one complete cycle, the energy loss is equal to:

The energy lost in completing one cycle of the *B-H* hysteresis loop

$$W_h = V \times \left(\text{area of } B\text{-}H \text{ loop}\right) \text{ J/cycle} \tag{5B.24}$$

This is a direct result of the hysteresis *B-H* relationship of the specimen.

### Eddy Currents

Consider a rectangular cross section core. If the magnetic field is periodically varying in time, then Faraday's Law tells us that a voltage will be induced in the ferromagnetic core, causing a current.

This results in a heat loss. The direction of the induced current will be such as to oppose the change in the flux (Lenz's Law).

Another mechanism of loss in a magnetic system is caused by induced currents in the iron

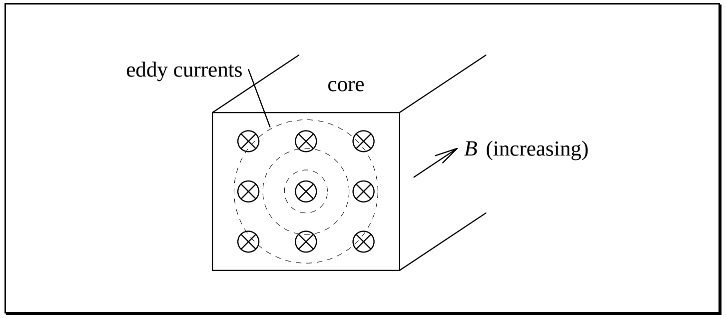



**Figure 5B.4**

*Explain why there is a non-uniform distribution of flux, and therefore an inefficiency in core usage. (Saturated at outer edges, not saturated in the centre). Label the direction of the eddy currents.*

To overcome this, the core is laminated to break up the eddy current paths. This results in lower losses (due to higher resistance) and better utilisation of the core area for the flux.

Eddy currents can be reduced by laminating the iron

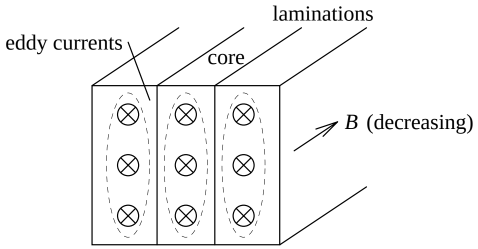



**Figure 5B.5**

The laminations are insulated either with an oxide layer or an enamel or varnish. This means the cross sectional area does not give us the cross sectional area of the ferromagnetic material – it includes the core insulation.

The cross sectional area of the laminations is not the same as that of the iron – take this into account with a "stacking factor"

We therefore define a stacking factor, which multiplies the cross sectional area to give the ferromagnetic cross sectional area. e.g. a stacking factor of 0.9 means 90% of the area is ferromagnetic, 10% is insulation.

If the rate of change of flux is large, a large voltage is induced, so eddy currents increase. If the frequency of an alternating flux is large, it tends to increase the area of the *B-H* loop.





## Summary

- The power delivered to an inductor is either dissipated as heat or used to store energy in the magnetic field. The stored energy in the magnetic field can be returned to the system.
- Co-energy is a quantity of energy that does not exist anywhere inside (or outside) a magnetic system; nevertheless, it is a useful quantity and will be seen to be related to the magnetic force.
- We can derive field energy density for both the electric and magnetic field.
- With ferromagnetic cores, when the applied field is varied, energy is dissipated as heat during the realignment of the domain walls and a power loss results. The area of the *B-H* hysteresis loop represents energy density loss in a magnetic system.
- If the magnetic field is periodically varying in time, then a voltage will be induced in any nearby ferromagnetic material, causing an eddy current. This results in a heat loss. To minimise this loss, we use laminated ferromagnetic cores.

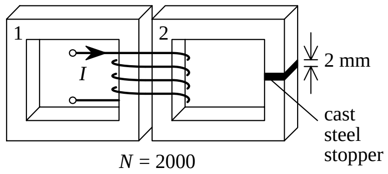
## References

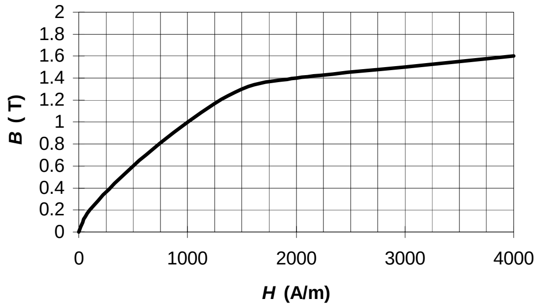
Plonus, Martin A.: *Applied Electromagnetics*, McGraw Hill Kogakusha, Ltd., Singapore, 1978.

## Problems

**1.**

The diagram shows two cast steel cores with the same csa $A = 5 \times 10^{-3}\ \text{m}^2$ and mean length $l = 500\ \text{mm}$.




(a) With DC excitation, the flux density in core 1 is $B_1 = 1.4\ \text{T}$. Determine $B_2$, $I$ and the self inductance of the circuit with and without the cast steel stopper in core 2.

(b) The cast steel stopper is removed and $I$ reduced to zero at a constant rate in 25 ms. Calculate the voltage induced in the coil.

(c) The exciting coil is now connected to a constant voltage, 50 Hz AC supply $\left(V = \hat{V}\cos\omega t\right)$. Determine $\hat{V}$, $\hat{I}$ and $\hat{B}_2$ if $\hat{B}_1 = 1\ \text{T}$ and the stopper is *in* core 2.

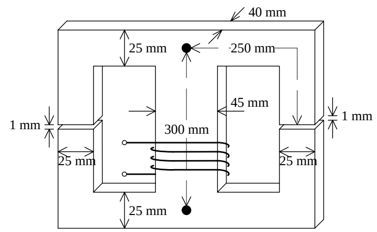
(d) Determine the field energy and the co-energy in core 2 for case (a).





**2.**

Determine the field energy and the inductance of the following circuit. The centre limb has a winding of 500 turns, carrying 1 A.




**3.**

A solenoid (diameter *d*, length *l*) is wound with an even number of layers (total number of turns $N_1$). A small circular coil ($N_2$ turns, area *A*) is mounted coaxially at the centre of the solenoid.

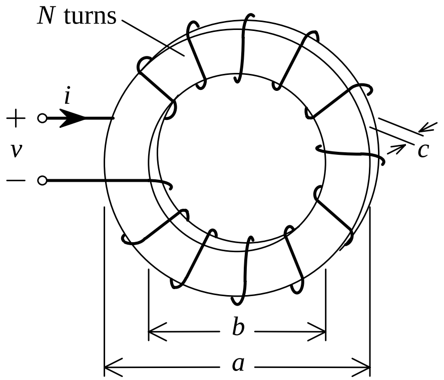
Show that the mutual inductance is:

$$M = \frac{\mu_0 N_1 N_2 A}{\sqrt{d^2 + l^2}}$$

Assume the field within the solenoid is uniform and equal to its axial field.

**4.**

Consider the toroid shown below:




$a$ = 160 mm, $b$ = 120 mm, $c$ = 30 mm

$\mu_r = 15000$, $N$ = 250

The exciter coil resistance is $r = 15\ \Omega$.

(a) Draw the electric equivalent circuit and determine the inductance $L$.

(b) An airgap ($l_g = 0.2$ mm) is cut across the core. Draw the magnetic and electric equivalent circuits, and determine the inductance representing the airgap. What is the electrical time constant of the circuit?





**5.**

Consider the toroid of Q4.

If $a$ = 80 mm, $b$ = 60 mm, $c$ = 20 mm, $N$ = 1500 and the normal magnetisation characteristic of the core is:

| $B$ (T) | 0.01 | 0.05 | 0.1 | 0.2 | 0.5 | 0.75 | 1.0 | 1.2 | 1.3 | 1.4 | 1.5 | 1.6 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $H$ ($\text{Am}^{-1}$) | 1.1 | 2.9 | 4.5 | 6.4 | 9.6 | 12.4 | 16 | 22 | 27.6 | 37 | 55 | 116 |

Plot:

(a) $\mu_r \sim B$ and $\mathcal{R} \sim B$ (a few points will be sufficient)

(b) normal inductance $L \sim i$

(c) Repeat (a) and (b) with a 0.5 mm airgap cut across the core.

**6.**

Show that the maximum energy that can be stored in a parallel-plate capacitor is $\varepsilon_r \varepsilon_0 E_b^2/2$ per unit volume ($E_b$ = maximum field strength before breakdown). Compare with the energy stored per unit volume of a lead acid battery.

Typical values:

| Material | Permittivity | Breakdown Field Strength |
|---|---|---|
| Air | $\varepsilon_r = 1$ | $E_b = 3$ kV/mm |
| Lead Acid | $\varepsilon_r = 3$ | $E_b = 150$ kV/mm |

# Lecture 6A – Rectification

*Full wave rectifier (FWR) circuits (centre-tapped transformer, bridge). Capacitor filter. Zener regulator.*

Due to the large (and ever increasing) amount of microelectronics used in electrical products, it is often desirable to have a steady DC supply that is derived from the AC "mains". To achieve this, it is necessary to somehow convert AC voltage into DC voltage. As a first step, we can convert the AC bipolar voltage into an AC unipolar voltage (one that still varies with time but always remains positive), a process known as rectification.

## Centre-tapped Transformer FWR Circuit

Consider the following circuit:

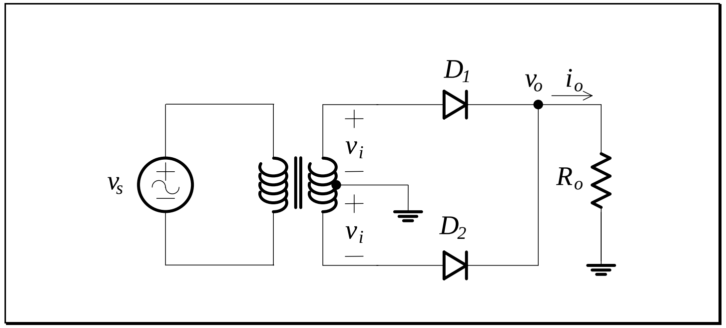



**Figure 6A.1**

The transformer steps down the supply voltage to the required value (e.g. from 240 V mains voltage to 9 V). The transformer is constructed so that the centre of the secondary winding is available as a terminal on the transformer. It is like two secondary windings joined in series. The two windings have a common terminal called the "centre tap".

We analyse the circuit in the usual way. Assume the diodes are ideal. Assume the diodes are acting like open circuits. We carry out an analysis. If the voltage across a diode is positive, then our assumption is incorrect.

For a resistive circuit, there are two cases that have to be considered – the supply positive and the supply negative. If there were a capacitor in the circuit, then we would consider four cases – positive increasing, positive decreasing, negative decreasing and negative increasing.

Firstly, assume the supply is positive and both diodes are off. After analysis, $D_1$ is found to be forward biased and therefore in the conducting state. KVL around the output circuit then gives:

$$v_o = v_i \tag{6A.1}$$

The output voltage equals the input voltage when the supply is positive. $D_2$ will remain in the off state, with a reverse bias voltage of magnitude $2v_i$ .

Now assume the supply is negative and both diodes are off. $D_2$ should conduct in this case. KVL then gives us:

$$v_o = -v_i \tag{6A.2}$$

In this case, $v_i$ is a negative number, so the output voltage will still be positive. $D_1$ will remain in the off state.

The result is a full wave rectifier:

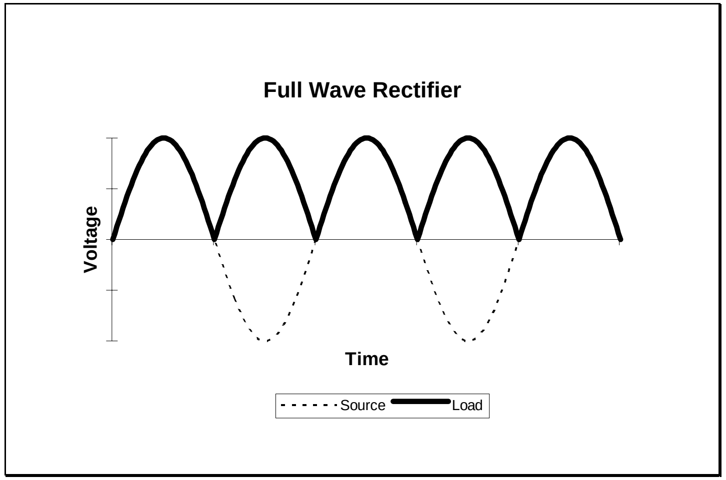



**Figure 6A. 2**





We can include the real effects of the diode, to a first approximation, by assuming a constant voltage drop model.

When a diode is off, the peak inverse voltage (PIV) it has to withstand can be obtained. This is important in selecting real diodes. In this case the PIV is $2\hat{v}_i$.

## Bridge FWR Circuit

The following circuit also acts as a FWR:

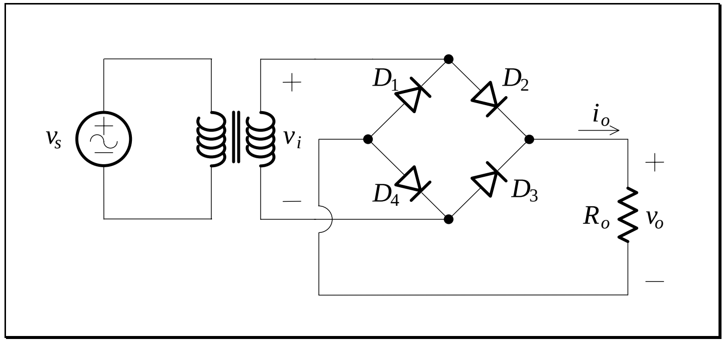



**Figure 6A.3**

We can perform the usual analysis quickly. In the positive half cycle $D_2$ and $D_4$ are on. $D_1$ and $D_3$ are effectively in parallel across the transformer secondary and are reverse biased. The PIV for each diode is $\hat{v}_i$. In the negative half cycle $D_1$ and $D_3$ are on, $D_2$ and $D_4$ are reverse biased. The output voltage is seen to be always positive.

The advantages of this configuration are: smaller transformer (no centre tap) and smaller PIV for each diode. The circuit is suitable for higher voltages.

## Capacitor Filter

The FWR voltage is not a good approximation to a DC voltage. The FWR waveform does have a steady DC value (the average of the waveform), but it still contains a large alternating component. To reduce this alternating component, we use a simple filter. We have seen the effect a switched capacitor makes on a varying voltage in Lecture 4A. It tends to smooth the waveform. If we connect a large capacitor across the output of the FWR, we get the following circuit:

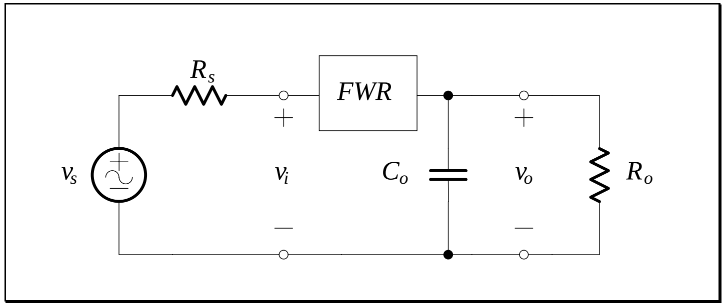



**Figure 6A.4**

An analysis of the above circuit (as was done in Lecture 4A) reveals that the output of the FWR with a capacitor is:

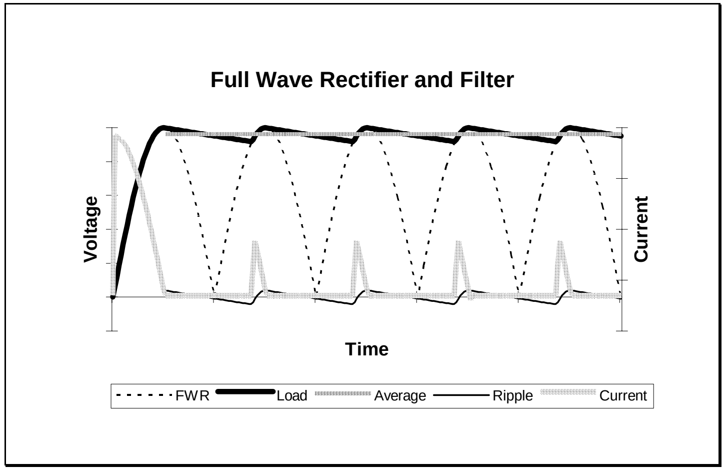



**Figure 6A.5**





If the time constant is large, then the exponential decay shown in the load voltage above can be approximated by:

$$v_o(t) \approx \hat{v}_o\left(1 - \frac{t}{R_o C_o}\right) \tag{6A.3}$$

which is the equation of a straight line.

Since the discharge time is much larger than the charging time, we can approximate this time by the period of the full wave rectified waveform, *T/2*.

The peak to peak excursions, or ripple, about the average value is then given by:

$$\begin{aligned} \delta v_o &= \hat{v}_o - \hat{v}_o\left(1 - T/2R_o C_o\right) \\ &= \frac{\hat{v}_o T}{2R_o C_o} = \frac{\hat{v}_o}{2fR_o C_o} \end{aligned} \tag{6A.4}$$

The average or DC voltage is therefore:

$$\begin{aligned} V_{DC} &= \hat{v}_o - \frac{\delta v_o}{2} \\ &= \hat{v}_o - \frac{\hat{v}_o}{4fR_o C_o} \\ &\approx \hat{v}_o - \frac{I_{DC}}{4fC_o} \end{aligned} \tag{6A.5}$$

The ripple is reduced compared to the HWR, simply because the period of the waveform has halved. The output is now approaching DC. We can further smooth this wave using a regulator.

## Zener Regulator

A Zener diode is a diode that exhibits Zener breakdown when it is reverse biased. Zener breakdown occurs when the electric field in the depletion layer is strong enough to generate hole-electron pairs, which are accelerated by the field. This increases the reverse bias current. It gives rise to a sharper transition and steeper curve than forward biased conduction.

The definition of regulation is the percentage change in output for a given input:

$$\text{regulation} = \frac{\text{change in output}}{\text{change in input}} \times 100\% \qquad (6A.6)$$

A reverse biased Zener diode can be added in parallel to the rectifier output.

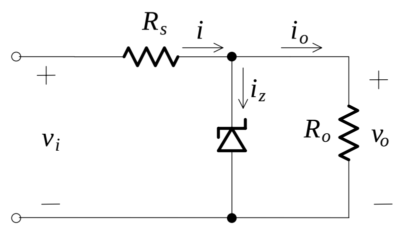



**Figure 6A. 6**

Large changes in input voltage or load current produce small voltage variations.

*Describe how the vertical characteristic of the Zener achieves regulation. Demonstrate with* 1 kΩ *resistors and* 5 mA *total current.*





## Summary

- Rectification is the process of converting a bipolar waveform to a unipolar waveform. A rectified waveform can have both a DC and an AC component.

- A full-wave rectifier can be created using a centre-tapped transformer and two diodes, or by using an ordinary transformer and a diode bridge. Each circuit has advantages and disadvantages.

- A capacitor is placed on the output of a full-wave rectifier to increase the DC component whilst also reducing the AC component (or ripple).

- A Zener diode can be used as a simple regulator by operating the diode in the reverse breakdown region.

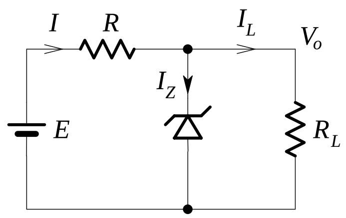
## References

Sedra, A. and Smith, K.: *Microelectronic Circuits*, Saunders College Publishing, New York, 1991.

## Problems

**1.**

In the circuit shown, the Zener diode "regulates" at 50 V for $5 \le I_Z \le 40\text{ mA}$, and $E = 200\text{ V}$.




(i) Determine the value of $R$ needed to allow voltage regulation $(V_o = V_Z)$ from a load current $I_L = 0$ to $I_{L\max}$.

(ii) With $R$ = value determined in (i) and $I_L = 25\text{ mA}$, determine $E_{\min}$ and $E_{\max}$ for regulation to be maintained.

# Lecture 6B – The Transformer Principle

*Transformer electric and magnetic equivalent circuits. Stray capacitance. Sign convention.*

## Transformer Electric and Magnetic Equivalent Circuits

A transformer can be as simple as the following arrangement:

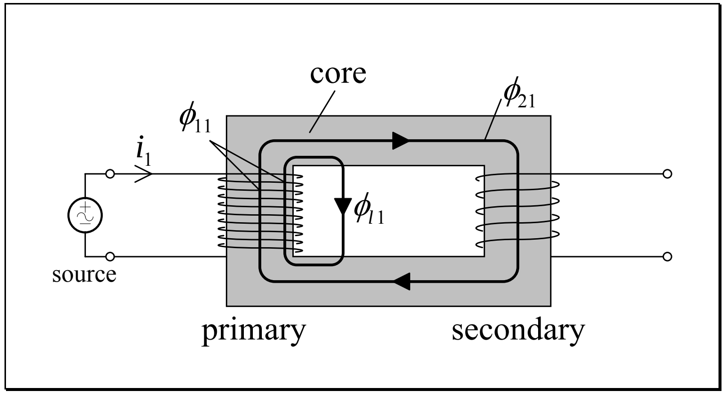



**Figure 6B.1**

It consists of a ferromagnetic core (to direct the flux along a particular path) and two windings. One winding is connected to a source, the other to a load. The source supplies current to the primary winding, which creates a flux in the core. Most of this flux streams through the core and links the secondary winding ($\phi_{21}$). Some of the flux leaks through the air ($\phi_{l1}$).

Suppose the source is a sinusoidally varying voltage. If the resistance of the winding is small, then the induced emf across the primary winding will equal the source voltage. *Show this using KVL.* Faraday's law then tells us that the flux will be sinusoidal.

Since a time varying flux is linking the secondary winding, there will be an emf induced in that winding. With no load connected to the secondary winding, this emf has no effect on the flux (the secondary is an open circuit, so there is no current). The transformer is just an iron cored inductor.

If we now connect a load to the secondary winding, the induced emf will force a sinusoidal current. The emf acts like an ordinary voltage source. Remember: an emf is not a passive voltage like that across a capacitor – there is something going on behind the scenes (chemistry for a battery, motion for a generator, Faraday's law for a transformer, etc) that allows it to supply a steady current.

The flux in the transformer, due entirely to the secondary winding carrying a current, is shown below for the case of $i_2$ coming out of the top secondary winding:

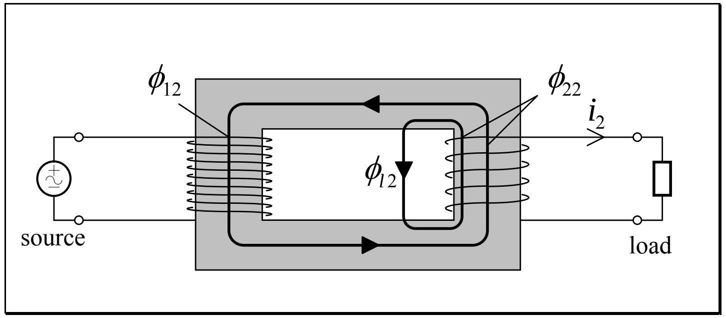



**Figure 6B.2**

We can now imagine the total flux streaming through the transformer:

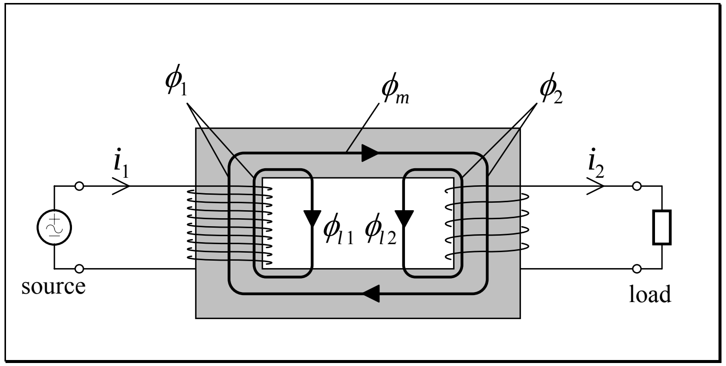



**Figure 6B.3**





*When do the directions of current shown above apply? What polarity do the induced emfs on the windings have? Can the load current be in the other direction?*

We can draw the following conclusions from the above picture:

$$\begin{aligned} \phi_m &= \phi_{21} - \phi_{12} \\ \phi_1 &= \phi_m + \phi_{l1} \\ \phi_2 &= \phi_m - \phi_{l2} \end{aligned} \tag{6B.1}$$

The flux linking the primary winding is $\phi_1$. The flux linking the secondary winding is $\phi_2$.

The total flux linkages of each winding are:

$$\begin{aligned} \lambda_1 &= N_1 \phi_1 \\ \lambda_2 &= N_2 \phi_2 \end{aligned} \tag{6B.2}$$

We will now apply KVL to each winding, taking into account the winding resistance and the induced emf. On the source side we have:

$$v_1 = R_1 i_1 + \frac{d\lambda_1}{dt} \tag{6B.3}$$

where $v_1$ is the source voltage. On the load side we have:

$$v_2 = -R_2 i_2 + \frac{d\lambda_2}{dt} \tag{6B.4}$$

where $v_2$ is the load voltage. *Verify that this is true and label the voltage polarities on the previous diagram. What is $R_2$?*

The emfs induced in the two windings can also be expressed by:

$$\frac{d\lambda_1}{dt} = N_1\frac{d\phi_1}{dt} = N_1\frac{d\phi_m}{dt} + N_1\frac{d\phi_{l1}}{dt} \tag{6B.5a}$$

$$\frac{d\lambda_2}{dt} = N_2\frac{d\phi_2}{dt} = N_2\frac{d\phi_m}{dt} - N_2\frac{d\phi_{l2}}{dt} \tag{6B.5b}$$

We can now see two distinct parts of the induced emf. One is associated with the leakage flux, the other with the mutual flux.

Let's define the leakage inductance of each winding:

$$L_{l1} = \frac{N_1\phi_{l1}}{i_1}, \qquad L_{l2} = \frac{N_2\phi_{l2}}{i_2} \tag{6B.6}$$

and the emfs induced in the windings by the mutual flux:

$$e_1 = N_1\frac{d\phi_m}{dt}, \qquad e_2 = N_2\frac{d\phi_m}{dt} \tag{6B.7}$$

From the above equations we can see that:

$$\frac{e_1}{N_1} = \frac{e_2}{N_2} \tag{6B.8}$$

This equation comes from Faraday's law, and is called a *volts per turn balance*.

It relates the primary and secondary voltages on a transformer using the turns ratio. The KVL equations can now be written:

$$v_1 = R_1 i_1 + L_{l1}\frac{di_1}{dt} + e_1 \tag{6B.9a}$$

$$v_2 = -R_2 i_2 - L_{l2}\frac{di_2}{dt} + e_2 \tag{6B.9b}$$





Our model of the transformer can now be represented by a core that exhibits no leakage, since we can explicitly show the leakage component of inductance in the electrical circuit:

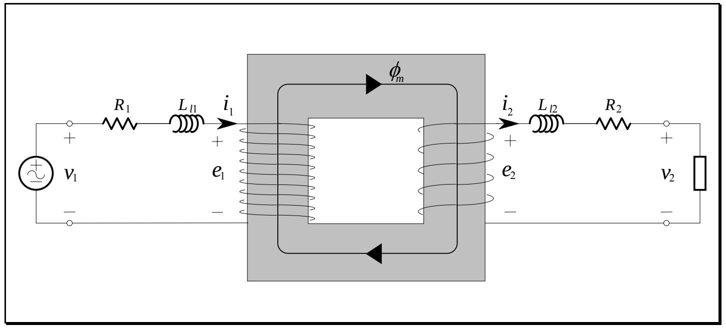



**Figure 6B.4**

Notice how the core has lost its leakage flux. We have transferred it so that it is represented in the electrical circuit, instead of the magnetic circuit.

*Verify that Eqs.* (6B.9a) *are KVL around the primary and secondary in the above model.*

The magnetic equivalent circuit of the arrangement above is therefore:

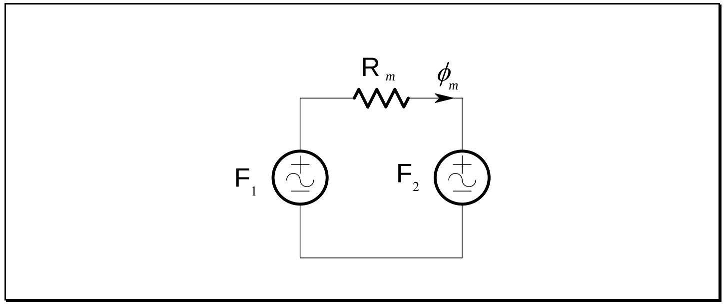



**Figure 6B.5**

We can reduce the magnetic circuit further so that there is only one mmf that creates the flux:

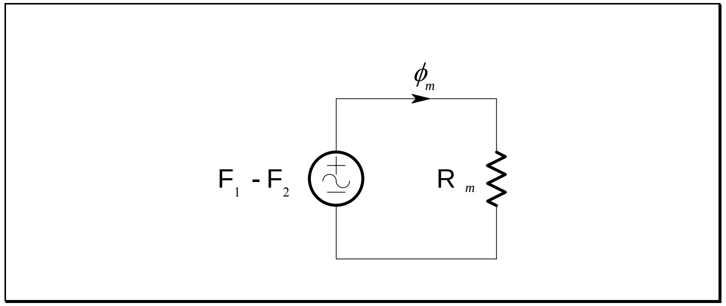



**Figure 6B.6**

Now imagine that this imaginary mmf was produced by the primary winding only. We can call this mmf a magnetising mmf since it is the cause of the flux in the core (it magnetises it).

The imaginary mmf is given by:

$$N_1 i_m = N_1 i_1 - N_2 i_2 \tag{6B.10}$$

where $i_m$ is called the *magnetizing current*. This equation is called an *mmf balance*, and gives the relationship between currents in a transformer. Since we imagine a magnetizing current producing a flux, it must be in an inductive circuit (inductors are the producers of flux). The above expression can be rearranged to give:

$$i_2' = \frac{N_2}{N_1} i_2$$
$$i_1 = i_m + i_2' \tag{6B.11}$$





This is KCL at the following node:

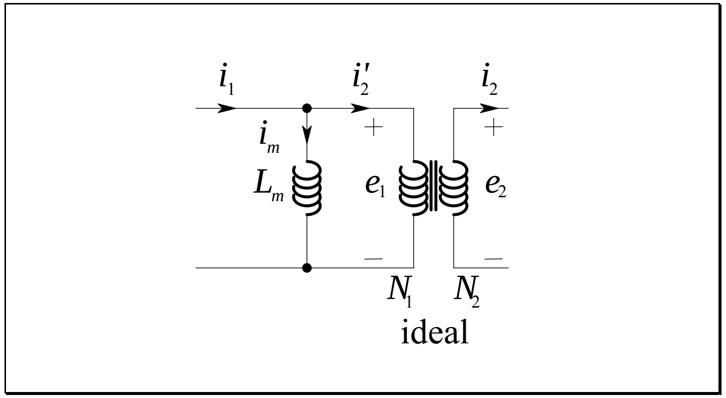



**Figure 6B.7**

The inductance $L_m$ is defined as:

$$L_m = \frac{N_1 \phi_m}{i_m} \tag{6B.12}$$

*Show why we define it to be this value. Hint: the inductor is in parallel with the ideal primary winding. Use Faraday's law.*

Our electrical equivalent circuit for a transformer is now:

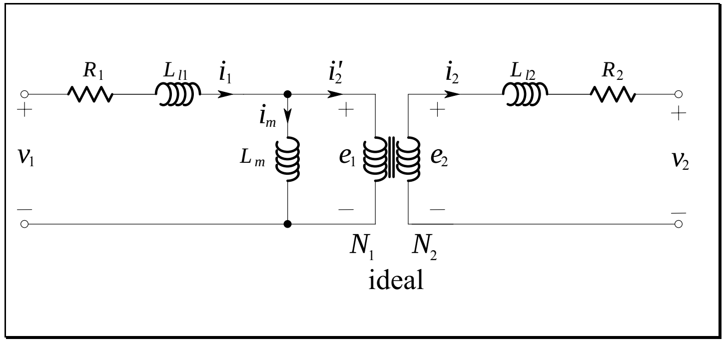



**Figure 6B.8**

The ideal transformer has relationships defined by Eqs. (6B.8) and (6B.11). We can "reflect" impedances from one side of the ideal transformer to the other. For example, let $\mathbf{Z}_2$ be the impedance seen by $\mathbf{E}_2$. Then the impedance seen by the primary is:

$$\mathbf{Z}'_2 = \frac{\mathbf{E}_1}{\mathbf{I}'_2} = \frac{N_1/N_2}{N_2/N_1}\frac{\mathbf{E}_2}{\mathbf{I}_2} = \frac{N_1^2}{N_2^2}\mathbf{Z}_2 \tag{6B.13}$$

We can replace the ideal transformer and everything on the load side by the equivalent impedance $\mathbf{Z}'_2$.

## Stray Capacitance

Stray interturn and interwinding capacitance affects the transformer's frequency response. At low frequencies (50 Hz) they are ignored.

## Sign Convention

A dot is placed on the ends of the primary and secondary winding of a transformer to indicate the polarity of the winding.

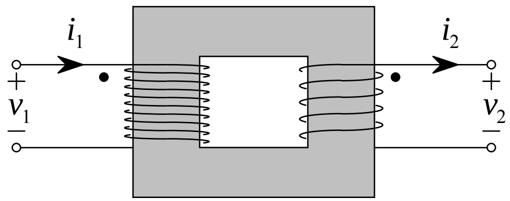



**Figure 6B.9**

At any instant:

- voltages on the windings have the same polarity with respect to the dot
- one winding has current into the dot, whilst the other has current out of the dot





## Summary

- A simple transformer consists of two windings wrapped around a ferromagnetic core. A sinusoidal source supplies current to the *primary* winding, which produces a changing flux in the core. This changing flux induces an emf in the *secondary* winding.
- Faraday's Law applied to a transformer leads to a relation known as a "volts per turn balance": $\frac{e_1}{N_1} = \frac{e_2}{N_2}$.
- In an ideal transformer with no magnetising current, Ampère's Law leads to a relation known as an "mmf balance": $N_1 i_1 = N_2 i_2$.
- The electrical equivalent circuit for a transformer has several elements: primary winding resistance $R_1$, primary leakage reactance $L_{l1}$, primary magnetising reactance $L_m$, and ideal transformer, secondary winding resistance $R_2$ and secondary leakage reactance $L_{l2}$.
- Approximations to the equivalent circuit of a transformer can be used in many cases to obtain quite reasonable models of transformer behaviour (it depends on the construction of the transformer and the external circuit).

## References

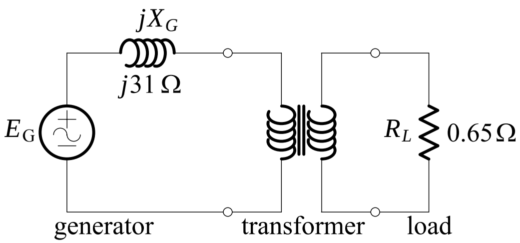

Slemon, G. and Straughen, A.: *Electric Machines*, Addison-Wesley Publishing Company, Inc., Sydney, 1982.

## Problems

**1.**

A 115 V RMS, 60 Hz generator supplies 3 kW to a $3\,\Omega$ load via a two winding transformer (assumed to be ideal). Determine the turns ratio and the minimum voltage and current ratings of each winding.

**2.**

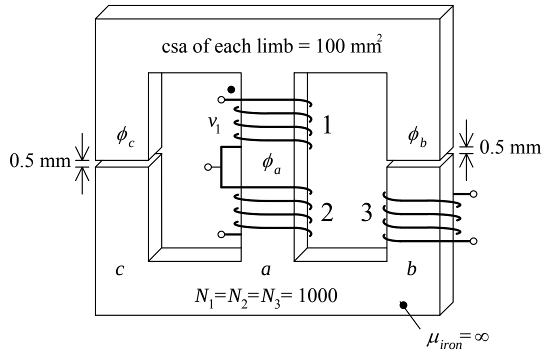



$E_G = 250\text{ V RMS}$, $f = 5\text{ kHz}$. Determine the transformer turns ratio needed to achieve maximum power into the load.





**3.**




Voltage $v_1$ is applied to winding 1. A current $i_1$ results.

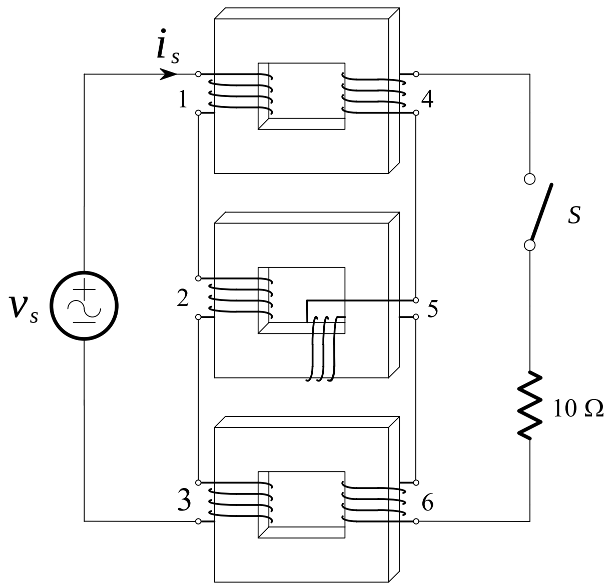
(a) Indicate the directions of $\phi_a$, $\phi_b$ and $\phi_c$. Place polarity markings (●) on windings 2 and 3.

(b) Draw magnetic and electric equivalent circuits. Calculate the mutual inductances $L_{21}$ and $L_{31}$.

(c) If $v_1 = \hat{v}_1 \cos(100\pi t)$, determine $\hat{v}_1$ and the peak magnetising current $\hat{i}_1$ needed to give $\hat{B}_a = 1\text{ T}$.

# 6B.12

**4.**




The three transformers are assumed ideal.

(a) Place appropriate polarity markings (●) on all windings. How many turns has winding 5?

(b) If $v_s = 30\sin(\omega t)$, determine the current in the $10\,\Omega$ resistor and $\hat{i}_s$ when switch $S$ is closed.

(c) What is the value of $\hat{v}_k$, $k = 1 \ldots 6$?

**<u>Hint</u>**: Use mmf and voltage / turn balance.

# Mid-Semester Revision

*Essential material (theory and problems). General knowledge topics.*

## Lecture 1A - essential

- Coulomb's Law
- Potential Difference
- Current, current density, Ohm's law
- Flux, flux density
- Gauss' law

You should be able to calculate field quantities for simple systems, e.g. coaxial cable, long thin wires.

## Lecture 1B - essential

- Law of Biot and Savart
- Magnetic potential
- Ampère's law
- Reluctance, magnetomotive force
- Lorentz force law
- Electromotive force
- Flux linkage
- Faraday's law
- Inductance

You should be able to calculate the inductance for simple systems, e.g. coaxial cable, two coaxial coils. You should be able to calculate the induced emf in a coil due to a changing flux linkage.

## Lecture 2A - general knowledge

- Polarization - essential
- Breakdown at sharp points - essential
- Air cavities in dielectrics - essential

You should be able to explain what the effect of a dielectric is when introduced into a capacitor, know why breakdown occurs at sharp points, and be able to predict the relative magnitudes of flux density and electric field in a system with more than one dielectric.

## Lecture 2B - general knowledge

## Lecture 3A - essential

- The *p-n* junction
- *p-n* junction characteristic
- Forward bias
- Reverse bias
- Linear model

You should be able to explain how a *p-n* junction works. You should be able to relate the *p-n* junction condition to its characteristic, and you should know how to model the diode with linear components.

## Lecture 3B - essential

- Curvilinear squares
- Coaxial cable

You should be able to calculate capacitance, inductance or resistance of arbitrary two dimensional conductor arrangements.





## Lecture 4A - essential

- Peak detector
- Clamp circuit
- Clipping circuit
- Graphical analysis

You should be able to explain the circuits and draw waveforms.

## Lecture 4B - essential

- Magnetic equivalent circuits
- Determine $\mathcal{F}$ given $\phi$
- Determine $\phi$ given $\mathcal{F}$

You should be able to apply Ampère's law and Gauss' law for any magnetic circuit containing a high permeability material.

## Lecture 5A - general knowledge

- Graphical analysis

Know that to analyze a nonlinear system, you have to use graphical techniques (or iterative numerical methods).

## Lecture 5B - essential

- Energy stored in a magnetic field
- Hysteresis loss

You should be able to derive the energy stored in a magnetic field. You should be able to explain hysteresis loss.

## Lecture 6A - essential

- Centre tap transformer full wave rectifier
- Bridge FWR
- Capacitor filter
- Zener regulator

You should be able to explain each circuit, and analyze each circuit to determine such things as PIV, ripple voltage, regulation.

## Lecture 6B - essential

- Electrical equivalent circuit

You should be able to explain what each component is meant to model, and how to measure them.

## General

All problems up to and including 6B should have been completed.

All examples in the notes should be read and understood.

# Lecture 7A - The MOSFET

*The metal oxide semiconductor field-effect transistor (MOSFET). Principle of operation. Output and transfer characteristics. Basic amplifier circuit (Q-point, biasing). Load line.*

## The MOSFET (*n*-channel)

The metal-oxide-semiconductor field-effect transistor (MOSFET) is the most widely used electronic device – it forms the basis of nearly every digital logic integrated circuit (IC), and is increasingly common in analog ICs.

There are two types of MOSFET – depletion-type and enhancement-type. The enhancement-type MOSFET is the mot widely used. An enhancement-type MOSFET is formed by the following process. Starting with a *p*-type *substrate*, two heavily doped *n*-type regions, called the *source* and *drain*, are created. A thin layer of silicon dioxide ($SiO_2$) is grown on the surface of the substrate between the drain and the source. Metal is deposited on top of the oxide layer, to from the *gate*. Metal contacts are also made to the source, drain and substrate (also known as the *body*). The device therefore has four terminals – gate (G), source (S), drain (D) and body (B).

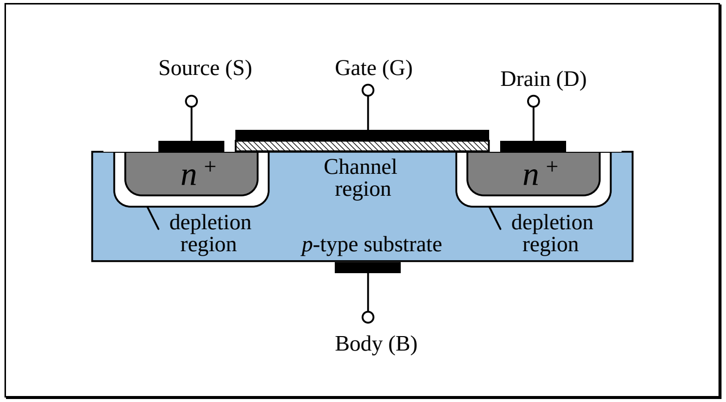



**Figure 7A.1 – MOSFET Device Structure**

The substrate forms a *p*-*n* junction with the source and drain regions, which are normally kept reverse-biased by connecting the substrate to the source. The MOSFET can then be treated as a three-terminal device.

A positive voltage applied to the gate will induce an *n*-type channel between drain and source (the channel is induced by the attraction of electrons from the heavily doped source and drain regions to the region beneath the positively charged gate electrode). This channel then allows conduction between the drain and the source, and the MOSFET is called an *n*-channel MOSFET, or an NMOS transistor.

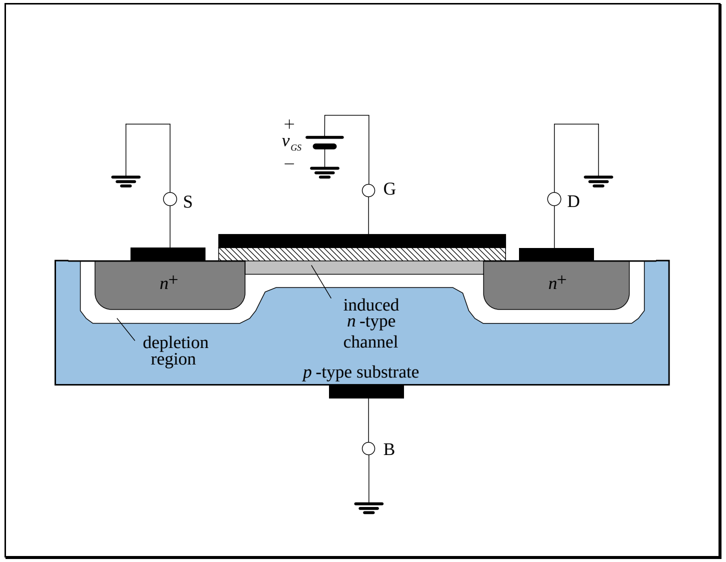



**Figure 7A.2 – MOSFET with Induced Channel**

The value of $v_{GS}$ at which a conducting channel is induced is called the *threshold voltage*, and is denoted by $V_t$. For an *n*-channel MOSFET, the threshold voltage is positive.

Once a channel is induced, if a small voltage $v_{DS}$ is applied between drain and source, then there will be a current between drain and source. The channel effectively looks like a resistor. If we increase the gate voltage, more charge





will be induced into the channel, the channel will grow, and the resistance will fall. Thus, the gate voltage controls the resistance of the channel, and hence the current between drain and source. This operation forms the basis of a transistor – one terminal of the device controls the current between the other two terminals.

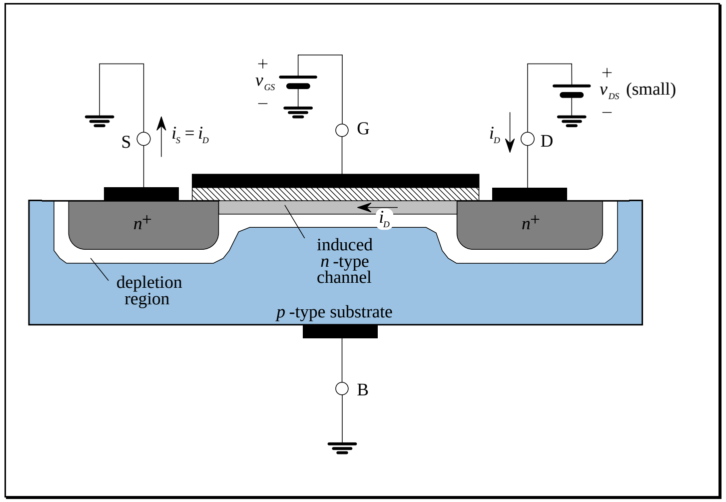



**Figure 7A.3**

As the voltage between the drain and source is increased, $v_{DS}$ appears as a voltage drop across the length of the channel. The channel now appears tapered. When $v_{DS}$ is increased to a value that reduces the voltage between the gate and the channel at the drain end to $V_t$, the channel depth decreases to almost zero and is said to be *pinched off*. Further increases in $v_{DS}$ cannot change the channel shape, and so the current in the channel remains constant at the value reached at pinch off. The MOSFET is said to enter the *saturation region*, and the voltage $v_{DS}$ at which this occurs is given by:

$$v_{DS\text{sat}} = v_{GS} - V_t \tag{7A.1}$$

It can be shown that in the saturation region, the relationship between drain current and gate-source voltage is given by:

$$i_D = K(v_{GS} - V_t)^2 \tag{7A.2}$$

where $K$ is a value that is dependent on the device construction.

## References

Sedra, A. and Smith, K.: *Microelectronic Circuits*, Saunders College Publishing, New York, 1991.

# Lecture 7B – The Transformer

*Magnetising branch. Voltage, flux and current waveforms. Phasor diagram. Losses and efficiency. Measurement of transformer parameters. Current and voltage excitation. 3rd harmonics.*

## Magnetising Branch

Our electrical equivalent circuit for a transformer derived previously was:

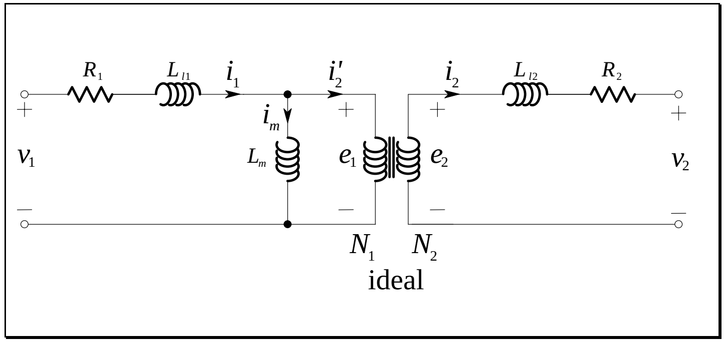



**Figure 7B.1**

This model assumed that the magnetizing inductance $L_m$ was linear. If we do not put a load on the secondary of the transformer, then $i_2$ will be zero. From the mmf balance equation, this also means that $i_2'$ will be zero – equivalent to an open circuit.

Therefore, on open circuit, our model reduces to that of an iron cored inductor:

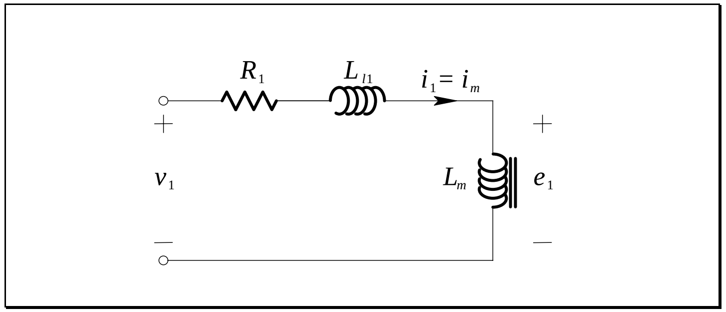



**Figure 7B.2**

We now want to take into account the *B-H* loop of the ferromagnetic core, instead of assuming that it is linear:

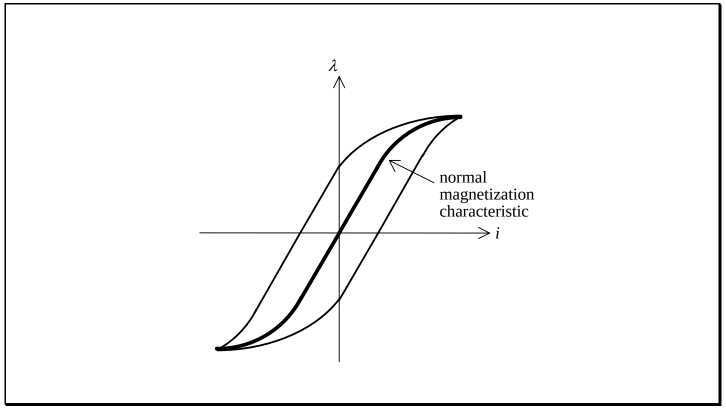



**Figure 7B.3**

We have seen that in traversing the *B-H* loop, we lose energy. To electrically model the *B-H* loop, we first reduce the area of the loop to zero, so that it has no losses. The *B-H* "loop" then reduces to the normal magnetization characteristic. Since the core is iron, eddy currents will be induced that will also contribute to the loss. To take into account all the losses, which we call *core losses*, we add a resistor in parallel to the magnetizing inductance. The total current, termed the *exciting current*, $i_e$, is therefore composed of a core loss component, $i_c$, and a magnetizing current, $i_m$:

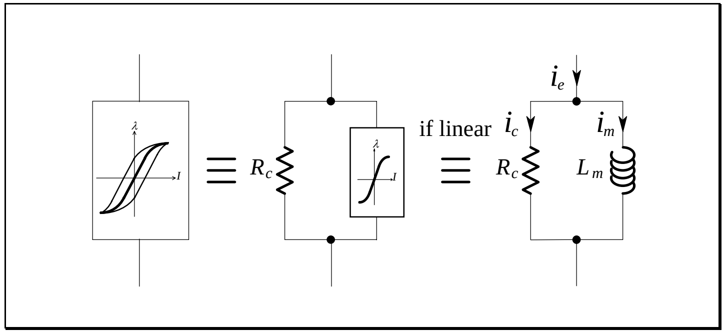



**Figure 7B.4**





If the transformer is operated over the linear region of the normal magnetization characteristic, then our equivalent circuit is all linear and represented by:

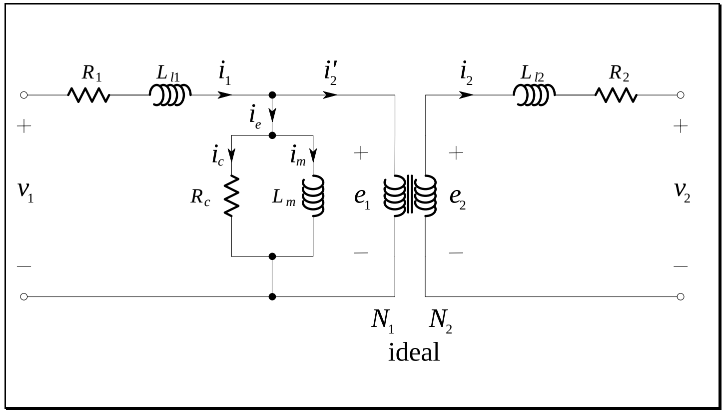



**Figure 7B.5**

## Voltage, Flux and Current Waveforms

Assume that the winding resistance and leakage inductance are small enough to be ignored. Then the source appears directly across the ideal transformer and magnetizing branch. Also, ignore the core loss. If we assume AC sinusoidal excitation, then we can use impedances and phasors in our electrical *frequency-domain* model of the transformer:

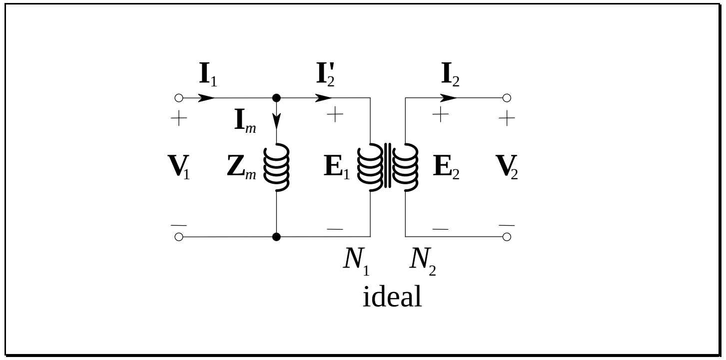



**Figure 7B.6**

The source is assumed to be sinusoidal. The flux in the core is given by Faraday's Law:

$$
\begin{aligned}
\phi &= \frac{1}{N}\int_0^t e_1 dt \\
&= \frac{1}{N}\int_0^t v_1 dt \\
&= \frac{1}{N}\int_0^t \hat{V}_1 \cos \omega t dt \\
&= \frac{\hat{V}_1}{N\omega}\sin \omega t
\end{aligned}
\qquad (7B.1)
$$

The flux therefore lags the voltage by 90º.

KCL at the input also gives:

$$
\begin{aligned}
i_1 &= i_m + i_2' = i_m + \frac{N_2}{N_1} i_2 \\
&= \hat{I}_m \sin \omega t + \frac{N_2}{N_1}\hat{I}_2 \cos(\omega t - \theta) \\
&= \hat{I}_1 \cos(\omega t - \beta)
\end{aligned}
\qquad (7B.2)
$$





The corresponding waveforms for the voltage, flux and current are:

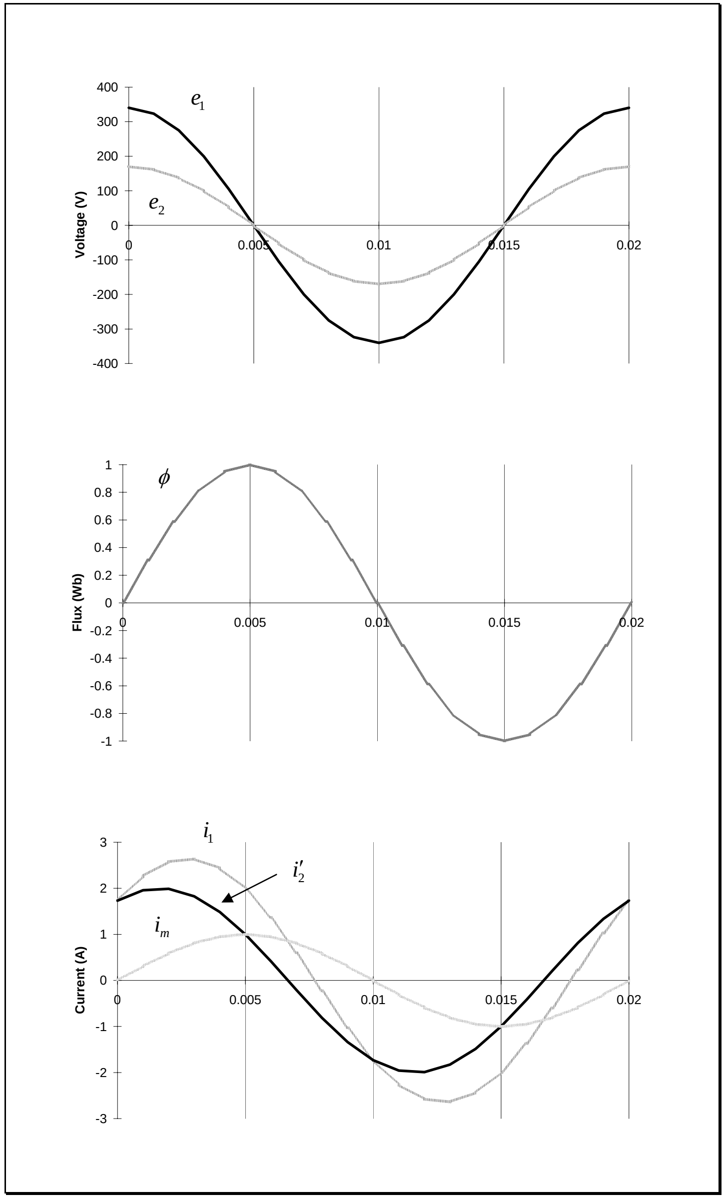



**Figure 7B.7**

## Phasor Diagram

Under sinusoidal excitation in the linear region of operation, the equivalent circuit of the transformer with all quantities referred to the primary side can be represented with phasor voltages and currents and impedances:

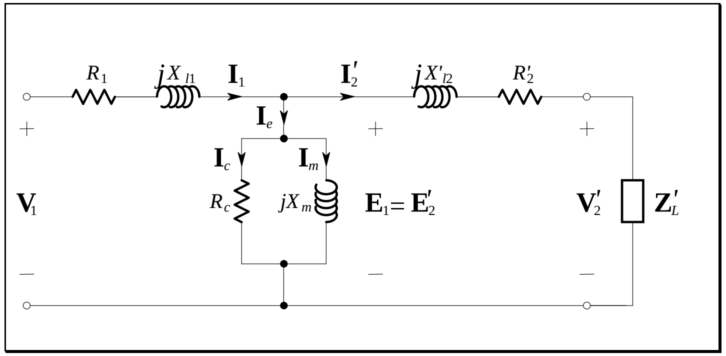



**Figure 7B.8**

Note that the variables and parameters on the secondary side of the transformer have been "referred" to the primary side of the transformer by using the relations derived earlier for the ideal transformer:

$$\mathbf{E}_2' = \frac{N_1}{N_2}\mathbf{E}_2 = \mathbf{E}_1$$

$$\mathbf{I}_2' = \frac{N_2}{N_1}\mathbf{I}_2$$

$$\mathbf{Z}_2' = \left(\frac{N_1}{N_2}\right)^2 \mathbf{Z}_2 \tag{7B.3}$$

The phasor diagram for the transformer is developed as follows. Draw $\mathbf{V}_2'$ as the reference. The current $\mathbf{I}_2'$ will be at some angle to this, depending upon the type of load. Assume a load circuit which possesses both resistance and inductance, and therefore operates at a lagging power factor.





Then KVL applied around the right-hand mesh of the transformer equivalent circuit gives:

$$\mathbf{E}_1 = \mathbf{E}_2' = \mathbf{V}_2' + \left(R_2' + jX_{l2}'\right)\mathbf{I}_2' \tag{7B.4}$$

Now we know $\mathbf{E}_1$, the exciting current can be determined by:

$$\mathbf{I}_e = \mathbf{I}_c + \mathbf{I}_m = \frac{\mathbf{E}_1}{R_c} + \frac{\mathbf{E}_1}{jX_m} \tag{7B.5}$$

Note that $\mathbf{I}_c$ is in phase with the voltage $\mathbf{E}_1$, and $\mathbf{I}_m$ lags $\mathbf{E}_1$ by 90º. The total primary current $\mathbf{I}_1$ can now be found from KCL:

$$\mathbf{I}_1 = \mathbf{I}_2' + \mathbf{I}_e \tag{7B.6}$$

Finally, KVL around the left-hand mesh gives:

$$\mathbf{V}_1 = \mathbf{E}_1 + \left(R_1 + jX_{l1}\right)\mathbf{I}_1 \tag{7B.7}$$

The complete phasor diagram showing all voltages and currents is:

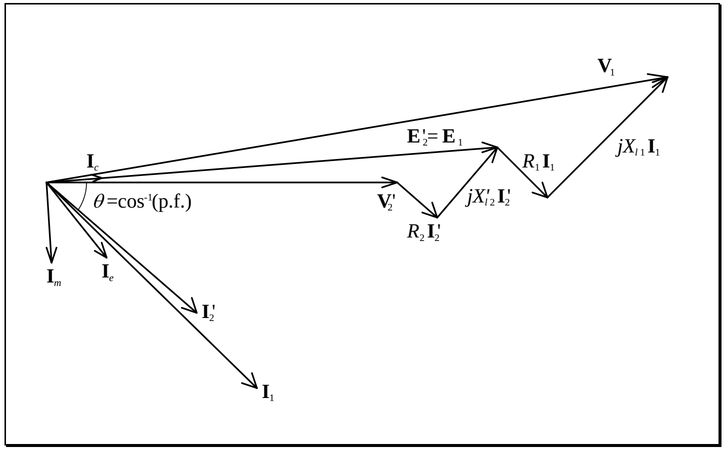



**Figure 7B.9**

**Example**

A 20-kVA, 2200:220-V, 50 Hz, single-phase transformer has the following equivalent-circuit parameters referred to the high-voltage side of the transformer:

$$R_1 = 2.51\,\Omega \qquad R_2' = 3.11\,\Omega$$
$$X_{l1} = 10.9\,\Omega \qquad X_{l2}' = 10.9\,\Omega$$
$$X_m = 25.1\,\text{k}\Omega$$

The transformer is supplying 15 kVA at 220 volts and a lagging power factor of 0.85. We would like to determine the required voltage at the primary of the transformer.

The equivalent circuit of Figure 7B.8 and phasor diagram of Figure 7B.9 are applicable, if we assume $R_c = 0$. Note that the transformer is not supplying its rated power. Also note that the rating gives the nominal ratio of terminal voltages – that is, it gives the turns ratio of the ideal transformer.

We proceed with the analysis as follows:

$$\mathbf{V}_2 = 220\angle 0°\ \text{V}$$
$$\mathbf{V}_2' = \frac{2200}{220}\mathbf{V}_2 = 2200\angle 0°\ \text{V}$$
$$|\mathbf{I}_2| = \frac{15\times 10^3}{220} = 68.2\ \text{A}$$
$$\cos^{-1} 0.85 = 31.7°$$
$$\mathbf{I}_2 = 68.2\angle -31.7°\ \text{A}$$
$$\mathbf{I}_2' = \frac{220}{2200}\times 68.2\angle -31.7° = 6.82\angle -31.7°\ \text{A}$$

We now have all the secondary variables and parameters referred to the primary side. From Figure 7B.8, we have:

$$\mathbf{E}_1 = \mathbf{V}_2' + \left(R_2' + jX_{l2}'\right)\mathbf{I}_2' = 2200\angle 0° + (3.11 + j10.9)6.82\angle -31.7° = 2260\angle 1.3°\ \text{V}$$
$$\mathbf{I}_m = \frac{\mathbf{E}_1}{jX_m} = \frac{2260\angle 1.3°}{25100\angle 90°} = 0.090\angle -88.7°\ \text{A}$$





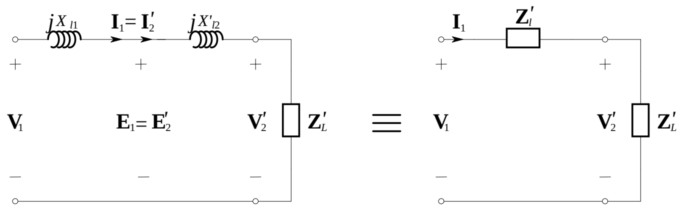
Note that $\mathbf{I}_m$ is very small compared with $\mathbf{I}_2'$. We now find the total primary current and voltage:

$$\mathbf{I}_1 = \mathbf{I}_m + \mathbf{I}_2' = 0.090\angle -88.7° + 6.82\angle -31.7° = 6.87\angle -32.3°\ \text{A}$$
$$\mathbf{V}_1 = \mathbf{E}_1 + \left(R_1 + jX_{l1}\right)\mathbf{I}_1 = 2260\angle 1.3° + (2.51 + j10.9)6.87\angle -32.3° = 2311\angle 2.6°\ \text{V}$$

Thus $|\mathbf{V}_1| = 2311\,\text{V}$, as compared with the rated or nameplate value of 2200 V. The additional voltage of 111 V is needed to "overcome" the impedance of the transformer.

Note also that the phasor diagram of Figure 7B.9 has greatly exaggerated the typical losses in a transformer for the sake of clarity of the drawing.

Also note that if we ignore the losses (resistances), and ignore the magnetizing current, we have a transformer model which looks like:




Analyzing, we get:

$$\mathbf{V}_1 = \mathbf{V}_2' + j\left(X_{l1} + X_{l2}'\right)\mathbf{I}_2' = 2200\angle 0° + j2\times 10.9\times 6.82\angle -31.7° = 2282\angle 3.2°\ \text{V}$$

This is not significantly different to the real voltage (a 1.3% error in terms of magnitude), which justifies modelling power transformers as just a leakage reactance under normal conditions of operation.

## Losses and Efficiency

### Iron Loss

This is the term used for any core loss – it includes hysteresis and eddy current losses. For most materials, the power loss is:

$$P_i \propto f\hat{B}^{1.6} \tag{7B.8}$$

It is independent of the load current.

### Copper Loss

This is the term used for heating loss due to the resistance of the windings.

$$P_c = R_1 I_1^{\,2} + R_2 I_2^{\,2} \tag{7B.9}$$

It is dependent upon the load current.

### Efficiency

Is a measure of how well a device converts its input to desired output. For a transformer, efficiency is defined as:

$$\begin{aligned}\eta &= \frac{\text{output power}}{\text{input power}} \times 100\% \\ &= \frac{\text{output power}}{\text{output power} + \text{losses}} \times 100\% \end{aligned} \tag{7B.10}$$





## Measurement of Transformer Parameters

### Open-Circuit Test

We leave the secondary of the transformer as on open circuit and apply the rated voltage on the primary side. With an open circuit on the secondary, we have already seen that the transformer is an iron-cored inductor:

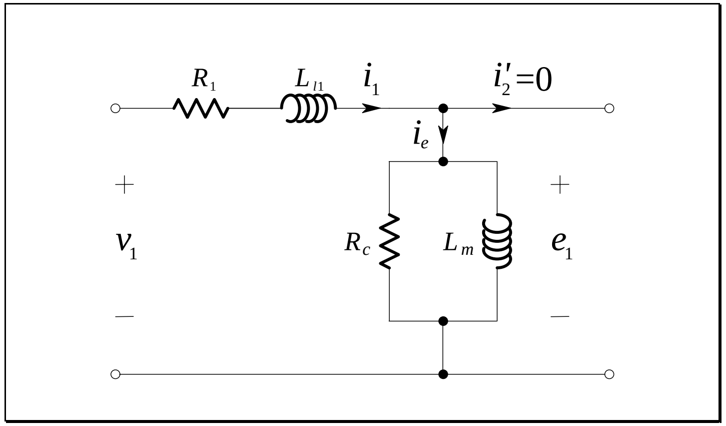



**Figure 7B.10**

The magnetizing branch has a higher impedance than the primary winding resistance and leakage inductance. They appear in series, so we can ignore the winding resistance and leakage inductance with a small error:

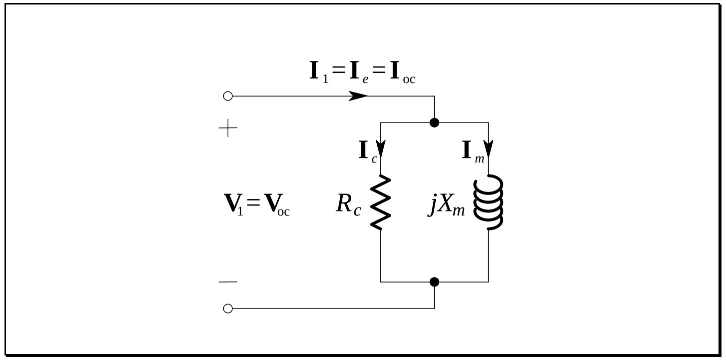



**Figure 7B.11**

The phasor diagram for this test is:

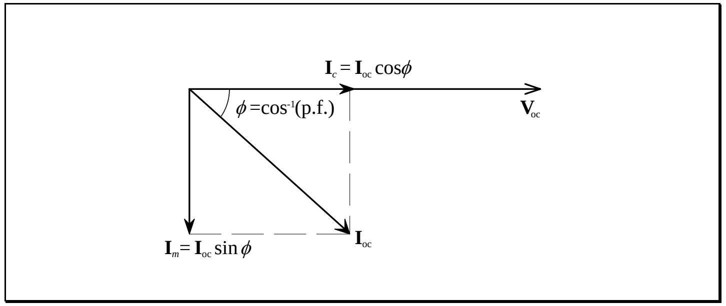



**Figure 7B.12**

If we measure the average power $P$, RMS voltage magnitude $|\mathbf{V}_{oc}|$ and RMS current magnitude $|\mathbf{I}_{oc}|$ then:

$$\cos\phi = \frac{P_{oc}}{|\mathbf{V}_{oc}||\mathbf{I}_{oc}|} \tag{7B.11a}$$

$$R_c = \frac{|\mathbf{V}_{oc}|}{|\mathbf{I}_{oc}|\cos\phi} \tag{7B.11b}$$

$$X_m = \frac{|\mathbf{V}_{oc}|}{|\mathbf{I}_{oc}|\sin\phi} \tag{7B.11c}$$

This gives us the magnetizing branch at rated voltage.





**Short-Circuit Test**

We apply a short circuit to the secondary of the transformer and increase the primary voltage until we achieve rated current in each winding. With a short on the secondary, we can reflect the secondary resistance and leakage inductance to the primary side:

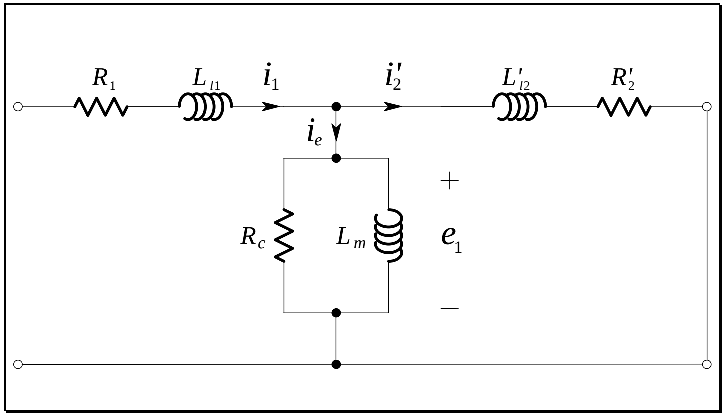



**Figure 7B.13**

The magnetizing branch has a higher impedance than the secondary impedance referred to the primary. They appear in parallel, so we can ignore the magnetizing branch with a small error:

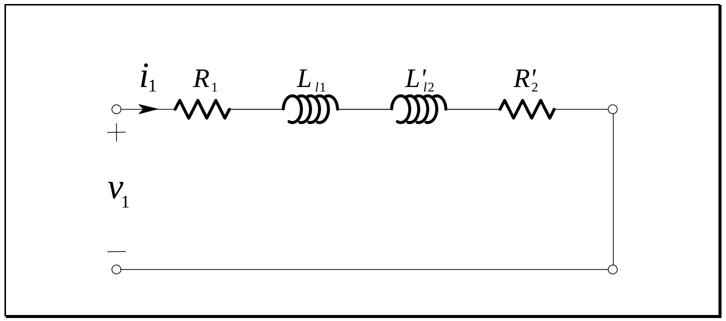



**Figure 7B.14**

The equivalent circuit for this test is effectively that shown below:

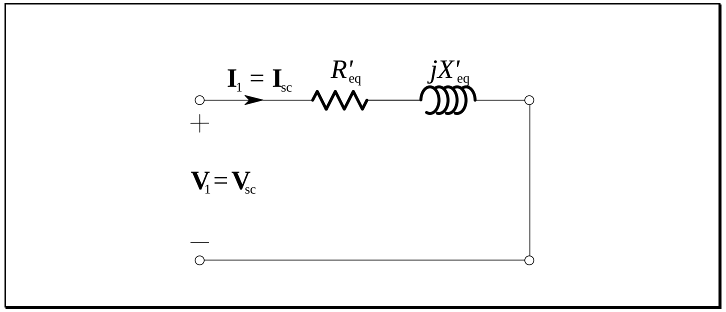



**Figure 7B.15**

where $R'_{\text{eq}}$ and $X'_{\text{eq}}$ are called the *equivalent resistance* and *equivalent leakage reactance* and are defined by:

$$R'_{\text{eq}} = R_1 + R'_2 \qquad \text{(7B.12a)}$$

$$X'_{\text{eq}} = X_{l1} + X'_{l2} \qquad \text{(7B.12b)}$$

It is then usually assumed (because the paths for the leakage flux of both windings are approximately the same) that:

$$X_{l1} = X'_{l2} = \frac{X'_{\text{eq}}}{2} \qquad \text{(7B.13)}$$





The phasor diagram for this test is:

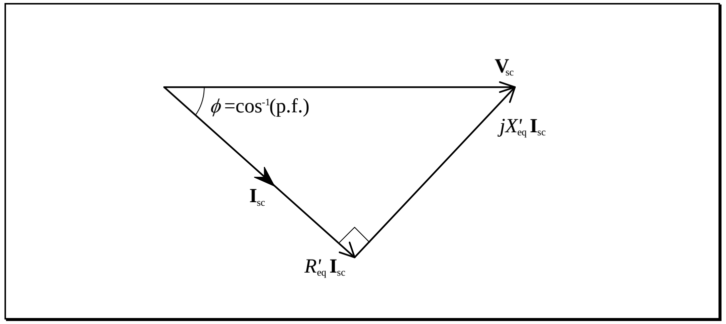



**Figure 7B.16**

If we measure the average power $P$, RMS voltage magnitude $|\mathbf{V}_{\text{sc}}|$ and RMS current magnitude $|\mathbf{I}_{\text{sc}}|$ then:

$$\cos\phi = \frac{P_{\text{sc}}}{|\mathbf{V}_{\text{sc}}||\mathbf{I}_{\text{sc}}|} \qquad \text{(7B.14a)}$$

$$R'_{\text{eq}} = \frac{|\mathbf{V}_{\text{sc}}|\cos\phi}{|\mathbf{I}_{\text{sc}}|} \qquad \text{(7B.14b)}$$

$$X'_{\text{eq}} = \frac{|\mathbf{V}_{\text{sc}}|\sin\phi}{|\mathbf{I}_{\text{sc}}|} \qquad \text{(7B.14c)}$$

### Winding-Resistance Measurements

$R_1$ and $R_2$ may be measured directly using a multimeter. Such measurements give the resistance of the windings to direct current, and it may be that these differ appreciably from the resistance to alternating current owing to non-uniform distribution of alternating currents in the conductors. This may be checked by determining $R'_{eq}$ from the short-circuit test and comparing it with the equivalent DC winding resistance referred to the primary side of the transformer. If $R_1$ and $R_2$ are the measured DC values, then the equivalent DC resistance is:

$$R'_{DC} = R_1 + \left( \frac{N_1}{N_2} \right)^2 R_2 \tag{7B.15}$$

Should $R'_{DC}$ differ appreciably from $R'_{eq}$, then the AC winding resistances referred to the primary side may be determined by dividing $R'_{eq}$ in the ratio of the two terms on the right-hand side of Eq. (7B.15).





## Current and Voltage Excitation

For any magnetic system with one electrical circuit (applies to transformer with open circuited secondary), KVL around the loop gives:

$$\begin{aligned} v &= Ri + e \\ e &= \frac{d\lambda}{dt} \end{aligned} \tag{7B.16}$$

There are two extreme cases we consider.

### Case 1 - Current Excitation

$R$ is large so that $e \approx 0$ (exact for DC). Then:

$$v \approx Ri \tag{7B.17}$$

and the system is said to be current excited. That is, we vary the voltage source which directly varies the current according to the above relationship. The flux in the system will then be determined from the $\lambda \sim i$ characteristic.

### Case 2 - Voltage Excitation

$R$ is small (applies to AC only) so that:

$$v \approx \frac{d\lambda}{dt} \tag{7B.18}$$

and the system is said to be voltage excited. That is, we vary the voltage source which directly varies the flux according to the above relationship. The current in the system will then be determined from the $\lambda \sim i$ characteristic.

## 3rd Harmonics

The non-linear $\lambda \sim i$ characteristic gives rise to unusual waveforms. Since the waveforms are periodic, we can make these strange waveforms by summating sine waves of different frequency, amplitude and phase. This is known as Fourier synthesis. To a close approximation, the magnetizing currents in most iron cores can be considered to be made of a fundamental (50 Hz) and a 3rd harmonic (150 Hz).

## References

Slemon, G. and Straughen, A.: *Electric Machines*, Addison-Wesley Publishing Company, Inc., Sydney, 1982.

# Lecture 8A – The MOSFET Voltage Amplifier

*Small signal equivalent circuit. The common-source amplifier. The common drain (or source follower) amplifier.*

## Small Signal Equivalent Circuit

When we looked at the MOSFET previously, we saw that the drain current $i_D$ was controlled by the voltage between the gate and source. The voltage $v_{GS}$ could increase or decrease the depth of the channel. There was a point where the current through the MOSFET could increase no further – this was termed saturation.

In real devices, saturation is not ideal. If we increase the applied voltage, $v_{DS}$, we also increase the drain current, $i_D$, by a small amount. The drain current, in the saturation region, is therefore dependent upon $v_{GS}$ and $v_{DS}$. For small variations in $v_{GS}$ and $v_{DS}$ we have:

$$\Delta i_D = \frac{\partial i_D}{\partial v_{GS}} \Delta v_{GS} + \frac{\partial i_D}{\partial v_{DS}} \Delta v_{DS} \tag{8A.1}$$

or:

$$\boxed{i_d = g_m v_{gs} + g_o v_{ds}} \tag{8A.2}$$

where:

$$\begin{aligned} g_m &= \text{slope of } i_D \sim v_{GS} \text{ characteristic at } Q\text{-point} \\ &= \text{transconductance} \end{aligned} \tag{8A.3a}$$

$$\begin{aligned} g_o &= \text{slope of } i_D \sim v_{DS} \text{ characteristic at } Q\text{-point} \\ &= \text{output conductance} \end{aligned} \tag{8A.3b}$$

Then for $g_m$ in Eq. (8A.3a), a graphical interpretation is:

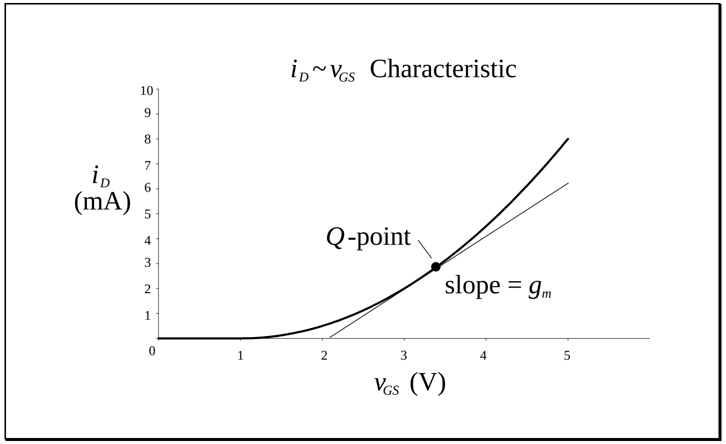



**Figure 8A.1**

and for $g_o$ in Eq. (8A.3b):

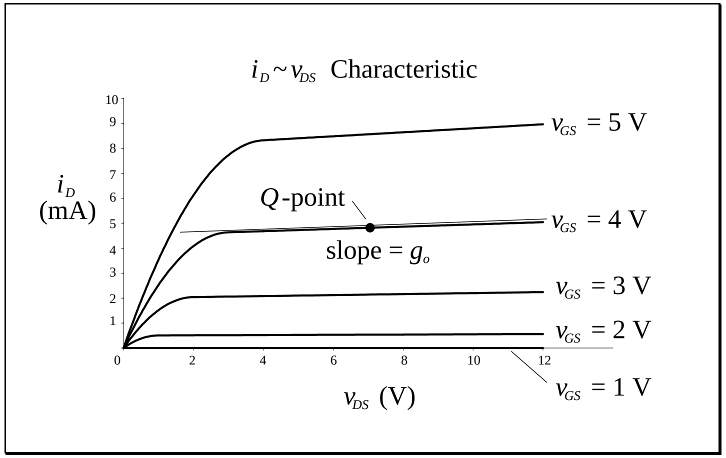



**Figure 8A.2**

If the signal is large, then the linear approximation given in Eq. (8A.2) is not very good, and we have to use a "large-signal" model of the MOSFET.





The formula of Eq. (8A.2) that relates small variations in $v_{GS}$ and $v_{DS}$ to $i_D$ can be put into circuit form. With small signals applied, the MOSFET characteristics look linear, and we can effectively model the MOSFET by an *AC small-signal equivalent circuit*:

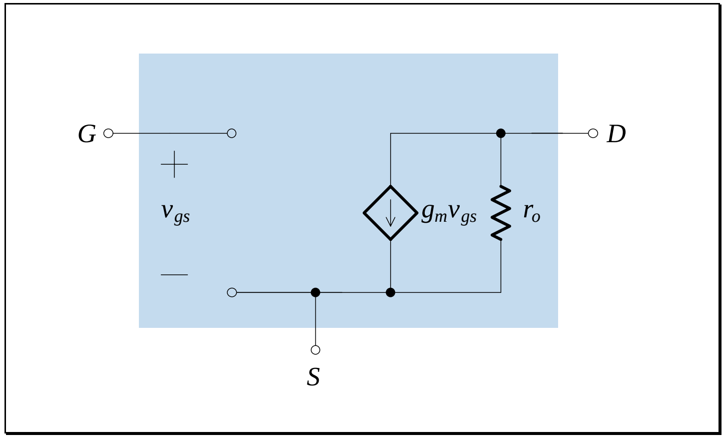



**Figure 8A.3**

*Confirm that an analysis of the above circuit will lead to Eq. (8A.2).* Since $r_o$ is large, it is common to ignore it in a *first-order hand analysis*. When this is the case, the equivalent circuit can be converted to a T equivalent-circuit:

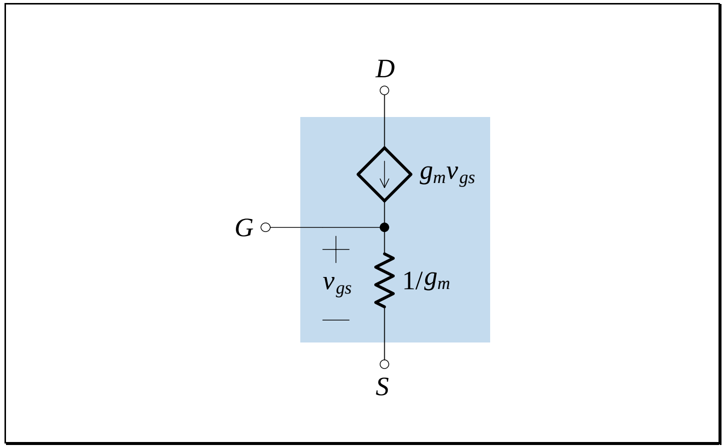



**Figure 8A.4**

Since there are depletion regions inside the MOSFET, there is charge separation. This results in a capacitance. These capacitances are due to the reverse-biased junctions and are of the order of 1 to 3 pF.

With small signals applied, the characteristic looks linear, and we can effectively model the MOSFET by an AC small signal equivalent circuit:

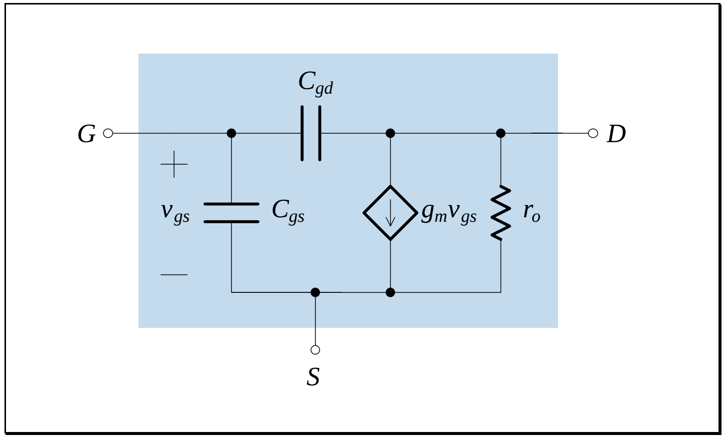



**Figure 8A.5**

The capacitors are ignored except when working at very high frequencies, where their reactance has an effect (> 100 kHz).

We can replace a MOSFET by its appropriate small-signal equivalent circuit when we perform an AC analysis of a properly DC-biased MOSFET.

We can now look at how to build various amplifiers using the MOSFET.





## The Common-Source Amplifier

The common-source amplifier is:

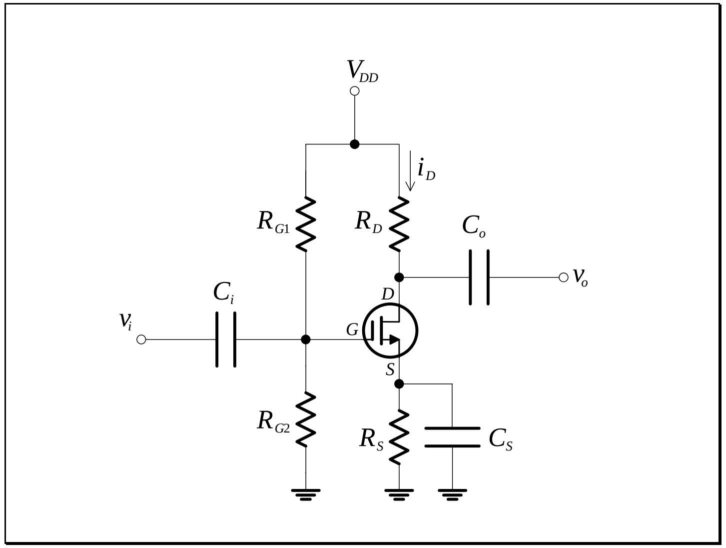



**Figure 8A.6**

The first part of analysing such a circuit is to determine the DC, or bias conditions on the MOSFET. The second part is to perform an AC analyis.

### DC Analysis

Note that for DC all capacitors are open circuits. This simplifies the DC analysis considerably.

Since the gate current is zero, the voltage at the gate is simply determined by the voltage divider formed by $R_{G1}$ and $R_{G2}$:

$$V_G = \frac{R_{G2}}{R_{G1} + R_{G2}} V_{DD} \tag{8A.4}$$

The voltage at the source is simply:

$$V_S = R_S I_D \tag{8A.5}$$

Thus, the gate-to-source voltage is:

$$V_{GS} = V_G - V_S = V_G - R_S I_D \tag{8A.6}$$

If this positive gate-to-source voltage exceeds the threshold voltage, the NMOS transistor will be turned on. We do not know, however, whether the transistor is operating in the saturation region or the triode region. If we assume it is operating in the saturation region, then to find the MOSFET's drain current, we can rearrange the Eq. (8A.6) to get the equation of a *load line*:

$$I_D = -\frac{1}{R_S}\left(V_{GS} - V_G\right) \tag{8A.7}$$

and graph it on the $i_D \sim v_{GS}$ characteristic:

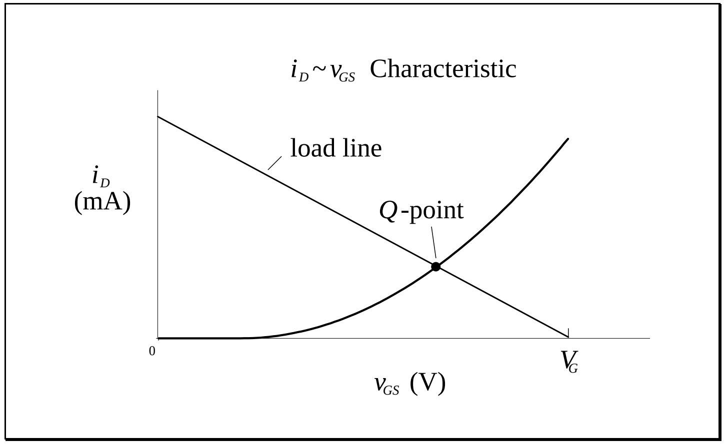



**Figure 8A.7**





Alternatively, we can find an algebraic solution:

$$\begin{aligned} I_D &= K\left(V_{GS} - V_t\right)^2 \\ &= K\left(V_G - R_S I_D - V_t\right)^2 \end{aligned} \tag{8A.8}$$

which results in the following quadratic equation:

$$R_S^2 I_D^2 + -\left[2K\left(V_G - V_t\right)R_S + 1\right]I_D + K\left(V_G - V_t\right)^2 = 0 \tag{8A.9}$$

Out of the two solutions to this quadratic, only one will intersect the $i_D \sim v_{GS}$ characteristic in the saturation region.

After finding $I_D$ we can then find the drain voltage:

$$V_D = V_{DD} - R_D I_D \tag{8A.10}$$

Lastly, we need to check that $V_D > V_G - V_t$, i.e. that the transistor is indeed operating in the saturation region.

**DC Design**

KVL, starting from the common and going up through the channel gives:

$$R_S I_D + V_{DS} + R_D I_D = V_{DD}$$

$$I_D = -\frac{1}{R_D + R_S}\left(V_{DS} - V_{DD}\right) \tag{8A.11}$$

This is the equation of a load line. The intersection of this line on the $i_D \sim v_{DS}$ characteristic gives the $Q$-point.

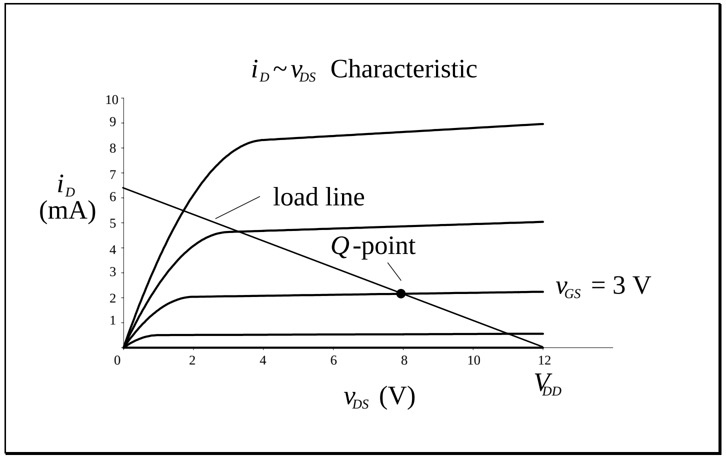



**Figure 8A.8**

When designing an amplifier, we use the rule of thumb that one-third of the power supply voltage, $V_{DD}$, appears across each of $R_D$, the transistor (i.e., $V_{DS}$) and $R_S$. This choice minimises variations in the $Q$-point as the threshold voltage of the MOSFET varies (from device to device), as well as providing roughly equal excursions of the output voltage when a small signal is applied.

It is possible to design the DC bias conditions algebraically by assuming that the transistor is in saturation, and using the applicable MOSFET voltage and current relations.





**AC Analysis**

AC analysis of electronic circuits relies on the fact that the circuit is linear and we can thus use superposition. Then the DC supply appears as a short circuit (i.e. an independent supply of 0 V). We also assume that the capacitors behave as perfect short circuits for the frequency at which our input signal is applied.

For small AC signals, the MOSFET small-signal equivalent circuit can be used in the AC analysis. We note also that the capacitor $C_S$ by-passes (effectively shorts) the resistor $R_S$. The DC supply is equivalent to a short circuit so that resistors $R_D$ and $R_{G1}$ are connected to common. The input and output capacitors $C_i$ and $C_o$ are used to couple the input and output voltages without disturbing the DC bias conditions, and are assumed to have zero reactance at "mid-band frequencies".

The AC small-signal equivalent circuit of the amplifier is then:

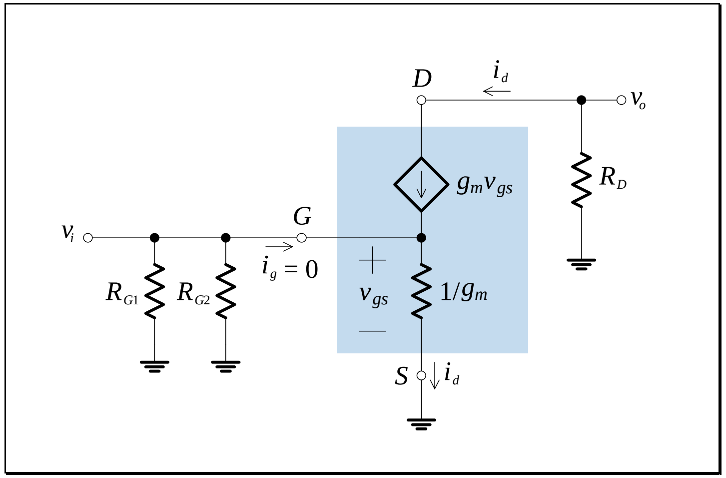



**Figure 8A.9**

Note that we have simply "dropped in" the small-signal equivalent circuit of the MOSFET – in this case the T equivalent.

We can now see that the input signal appears directly across the gate-to-source junction of the transistor:

$$v_{gs} = v_i \tag{8A.12}$$

and that the drain current is:

$$i_d = g_m v_{gs} \tag{8A.13}$$

The output voltage can now be found from:

$$\begin{aligned} v_o &= -i_d R_d \\ &= -g_m R_d v_i \end{aligned} \tag{8A.14}$$

What features of this circuit are we interested in? We are trying to create a voltage amplifier, so the quantities of interest are: the input impedance, the open circuit voltage gain, and the output impedance.

Thus the open-circuit (no load) voltage gain is:

$$A_{vo} = \frac{v_o}{v_i} = -g_m R_d \tag{8A.15}$$

The input resistance, as seen by the signal source, is, by inspection:

$$R_{\text{in}} = R_{G1} \parallel R_{G2} \tag{8A.16}$$

The output resistance is obtained by setting the input signal source to zero and observing the resistance "seen looking back into the output terminal":

$$R_{\text{out}} = R_D \tag{8A.17}$$

This value of output resistance is quite large, and is unsuitable in many applications.





We can now draw an equivalent circuit of the voltage amplifier:

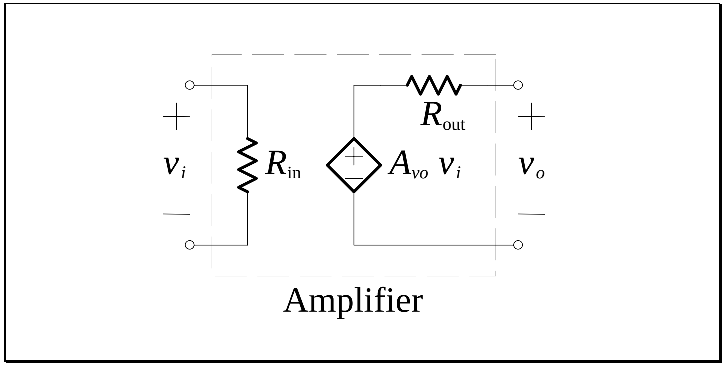



**Figure 8A.10**

This equivalent circuit can be used to study the performance of the amplifier when a real source (with input resistance) and a real load are connected:

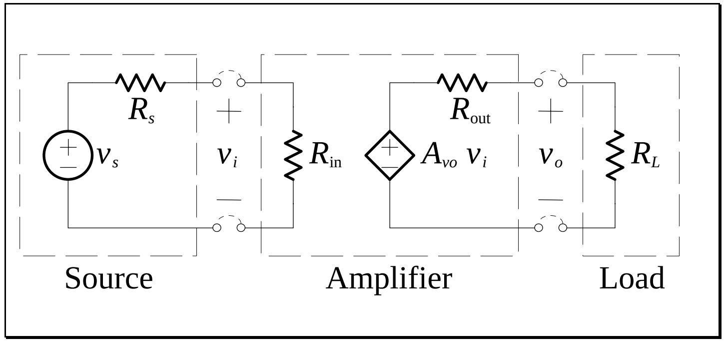



**Figure 8A.11**

A quick analysis shows that:

$$A_v = \frac{v_o}{v_s} = A_{vo} \frac{R_L}{R_L + R_{\text{out}}} \frac{R_{\text{in}}}{R_{\text{in}} + R_s} \tag{8A.18}$$

**AC Design**

As can be seen from Eqs. (8A.15) to (8A.17), the design of the amplifier from the small signal perspective is concerned with choosing a suitable drain resistance, $R_D$, and bias point $g_m$.

## The Common Drain (or Source Follower) Amplifier

The common-drain amplifier is:

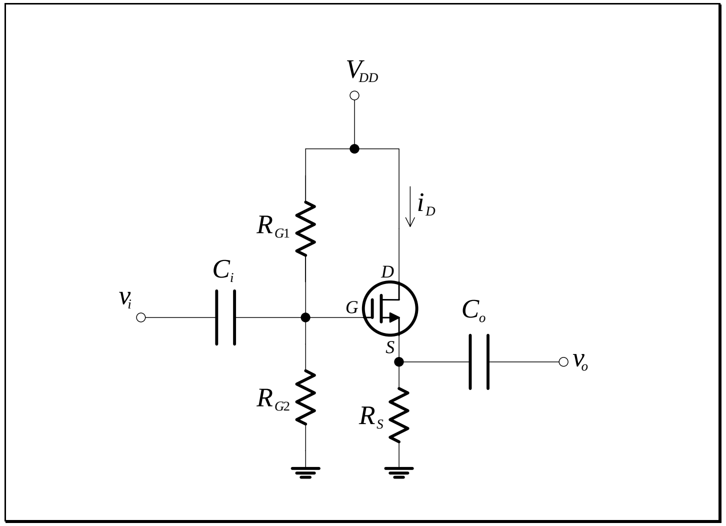



**Figure 8A.12**

Here, the output is taken from the source terminal, and the drain, for AC is connected to the common.





The small signal equivalent circuit for the common-drain amplifier is:

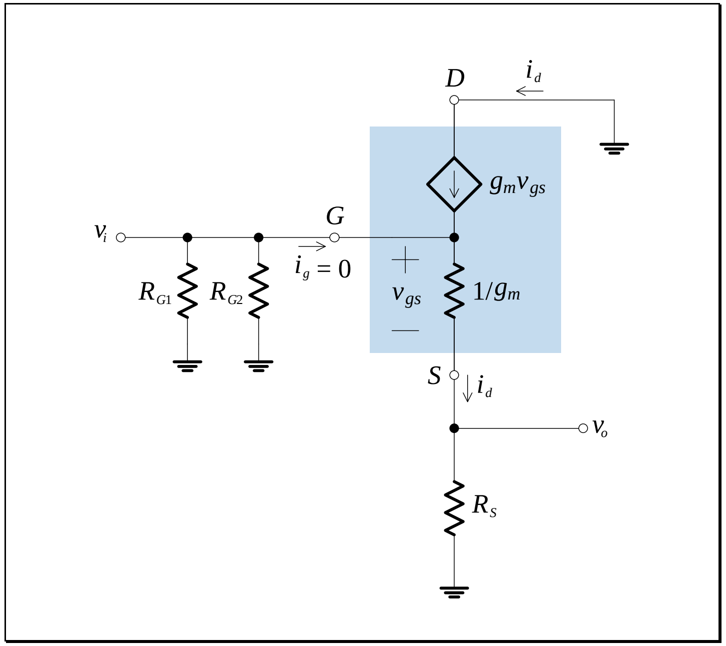



**Figure 8A.13**

Analysis of the circuit is straightforward:

$$v_o = \frac{R_S}{R_S + \frac{1}{g_m}} v_i \tag{8A.19}$$

which gives:

$$A_{vo} = \frac{v_o}{v_i} = \frac{R_S}{R_S + \frac{1}{g_m}} \tag{8A.20}$$

# 8A.14

Normally, $R_S >> 1/g_m$, causing the open-circuit voltage gain to be approximately unity. Thus the voltage at the source follows that at the gate, giving the circuit its popular name of *source follower*.

The input resistance is:

$$R_{\text{in}} = R_{G1} \| R_{G2} \tag{8A.21}$$

and the output resistance is:

$$R_{\text{out}} = R_S \| \frac{1}{g_m} \tag{8A.22}$$

Normally, $R_S >> 1/g_m$, which means $R_{out} \approx 1/g_m$ is moderately low. Thus the common-drain amplifier presents a low output resistance on its output and it can be used a unity-gain voltage buffer amplifier.

## References

Sedra, A. and Smith, K.: *Microelectronic Circuits, 5th Ed.*, Oxford University Press, New York, 2004.

# Lecture 8B – The Force Equation

*Force equation of singly excited electromechanical transducer. Electric field transducer. Electrostatic voltmeter.*

## Force Equation of Singly Excited Electromechanical Transducer

**Demo**

*Discuss arrangement of contactor. Ask what will happen with DC and AC excitation. Demonstrate. Uses - contactors, meters, motors.*

A simple electromechanical transducer could have the following arrangement:

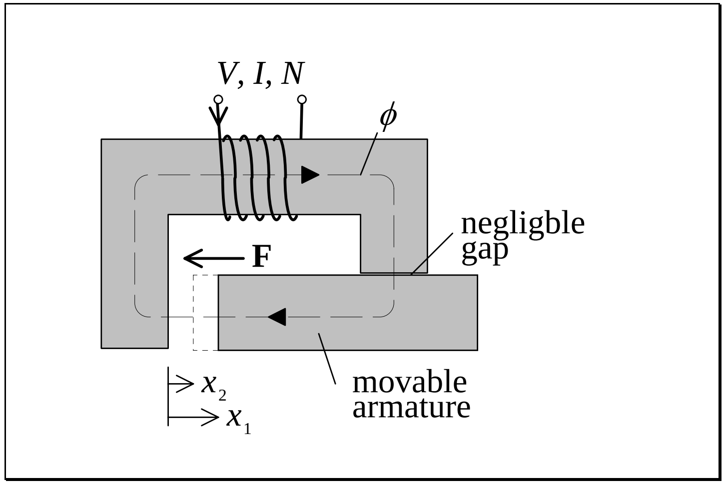



**Figure 8B.1**

It consists of a movable armature that is part of the core and one winding connected to a source. The source supplies current to the winding, which creates a flux in the core. The flux streams through the core and across the air gap.

The magnetic and electric equivalent circuits for the above arrangement are:

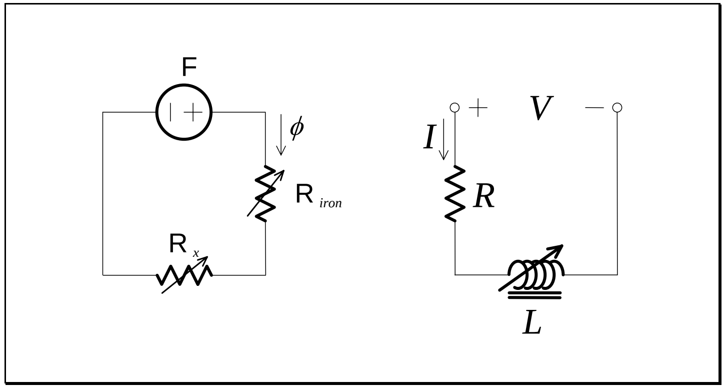



**Figure 8B.2**

The composite characteristic is almost linear due to the air gap. The characteristic varies as the distance *x* varies, since the gap width (reluctance, hence flux for a constant mmf) is changing.

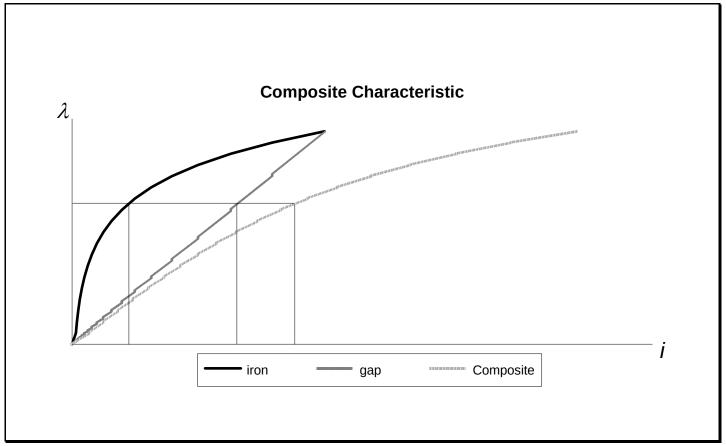



**Figure 8B.3**





Conservation of energy for the above system gives us:

| | | | | | |
|---|---|---|---|---|---|
| Electrical energy | | Mechanical energy | | Field energy | (8B.1) |
| minus | = | plus | + | plus | |
| resistance losses | | friction losses | | core losses | |

Expressed in symbols, this becomes:

$$W_e = W_m + W_f \tag{8B.2}$$

Suppose now that we are exciting the coil with a DC source. The steady state value of current is determined only by the resistance.

Electromechanical energy conversion occurs when the energy stored in the coupling field depends upon position. The armature can be imagined to have a north pole, while the fixed core has a south pole. The armature will be attracted to the fixed core.

If the armature moves slowly from position 1 to 2, the reluctance of the magnetic system is decreased. Since the mmf is constant, the flux must increase. To increase the flux, some energy is supplied to the magnetic field.

We assume the armature moves slowly in this case so that the emf produced by the changing flux is small and does not reduce the current. i.e. the current is constant.

In the time it takes the armature to move, an amount of electrical energy is supplied to the system:

$$\begin{aligned}\delta W_e &= VI\delta t - RI^2\delta t = \left(V - RI\right)I\delta t = eI\delta t \\ &= I\delta\lambda = I^2\delta L\end{aligned} \tag{8B.3}$$

We also remember that the amount of energy actually stored by the field in this time, for a linear lossless system is:

$$\delta W_f = \frac{1}{2} I \delta\lambda = \frac{1}{2} I^2 \delta L \qquad \text{(8B.4)}$$

Assuming a lossless system (no core losses) is a good approximation because the losses are usually very small. According to our energy conservation relation, any energy delivered electrically and not stored in the field produces mechanical work:

$$\begin{aligned} \delta W_m &= \delta W_e - \delta W_f \\ &= I^2 \delta L - \frac{1}{2} I^2 \delta L \\ &= \frac{1}{2} I^2 \delta L \end{aligned} \qquad \text{(8B.5)}$$

The work is done by the mechanical force that moves the armature through the distance $\delta x$:

$$\begin{aligned} \delta W_m &= F \delta x \\ F &= \frac{\delta W_m}{\delta x} \\ &= \frac{I^2}{2} \frac{\delta L}{\delta x} \end{aligned} \qquad \text{(8B.6)}$$





Again, mechanical losses such as friction have been ignored, because they are small. In the limit, as $\delta x$ becomes infinitesimally small:

$$F = \frac{I^2}{2} \frac{dL}{dx} = -\frac{\phi^2}{2} \frac{d\mathcal{R}}{dx} \qquad \text{(8B.7)}$$

since

$$L = \frac{N^2}{\mathcal{R}} \quad \text{and} \quad \phi^2 = \frac{N^2 I^2}{\mathcal{R}^2}$$

This equation was derived for the DC case but it also applies to the AC case.

The force will act in a direction so as to increase the inductance of the system.

## Electric Field Transducer

The electrical analogy to the previous magnetic system is the following system (the principle of the electrostatic speaker):

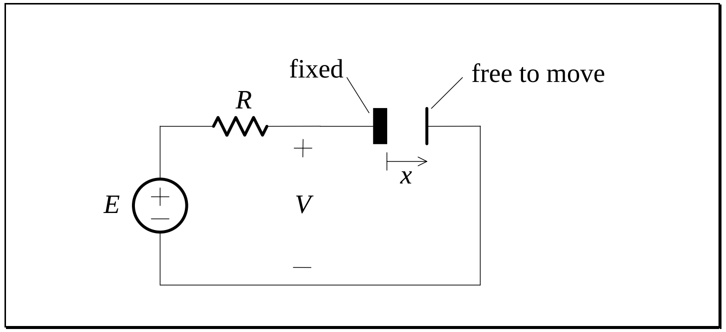



**Figure 8B.4**

The principle of operation is equivalent to the electromechanical device, except now the electric field is dominant – the magnetic field energy is ignored. This circuit is the dual of the electrical equivalent circuit for the electromechanical case.

We can replace *I* with *V*, and *L* with *C* to obtain:

$$F = \frac{V^2}{2}\frac{dC}{dx} \tag{8B.8}$$

The force will act in a direction so as to increase the capacitance of the system.

### Electrostatic Voltmeter

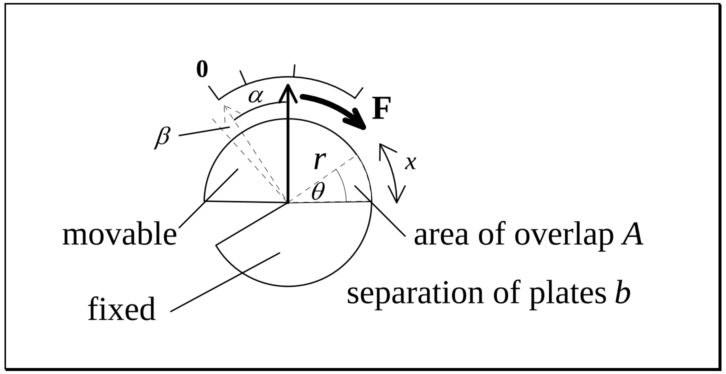



**Figure 8B.5**

The electric force equation applies. The force will tend to increase the capacitance, so $\theta$ increases.

The magnitude of the torque experienced by the movable plate is:

$$T = Fr = r\frac{V^2}{2}\frac{dC}{dx} = r\frac{V^2}{2}\frac{dC}{d\theta}\frac{d\theta}{dx} \tag{8B.9a}$$

$$x = r\theta, \quad \therefore \quad \frac{d\theta}{dx} = \frac{1}{r} \tag{8B.9b}$$

$$T = \frac{V^2}{2}\frac{dC}{d\theta} \tag{8B.9c}$$

i.e. in the force equation, replace *F*, *x* with *T*, $\theta$.





If fringing is ignored:

$$C = \frac{\varepsilon_0 A}{b} = \frac{\varepsilon_0}{b}\frac{r^2\theta}{2} \tag{8B.10}$$

$$\frac{dC}{d\theta} = \frac{\varepsilon_0}{b}\frac{r^2}{2} = 2K_d$$

The electrical deflecting torque is:

$$T_d = K_d V^2 \tag{8B.11}$$

The meter responds to the square of the voltage.

To operate as a meter, the movement of the plate is restrained by a spring that has a constant torsional restraint:

$$T_r = K_r \alpha \tag{8B.12}$$

When the two opposing torques balance, the needle will be steady on the scale, and we know that:

$$T_r = T_d \tag{8B.13}$$

$$K_r\alpha = K_d V^2$$

$$\alpha = KV^2$$

The meter therefore has a square law (nonlinear) scale.

### References

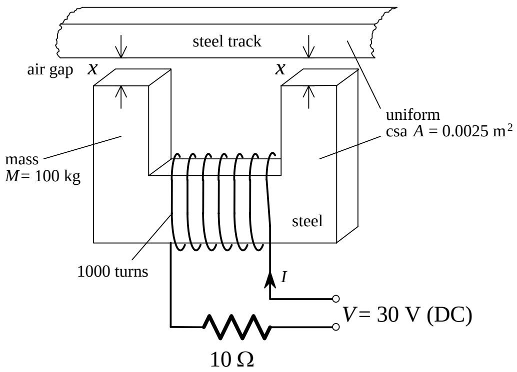
Slemon, G. and Straughen, A.: *Electric Machines*, Addison-Wesley Publishing Company, Inc., Sydney, 1982.

## Problems

### 1. [Magnetic suspension]

The device shown in the diagram consists of a steel track and a U-shaped bottom piece both of the same magnetic material, which can be considered ideal.

The bottom piece is prevented from falling by the magnetic field produced by current *I*.




(a) Draw the equivalent magnetic circuit and obtain an expression for the force *F* holding the bottom piece in terms of *x* (assume $\mu_{steel} = \infty$).

(b) At what value of *x* is *M* supported? Is the position stable? Sketch *F*~*x*. Give full explanations.

(c) How would the results of (a) and (b) be affected if fringing were considered?

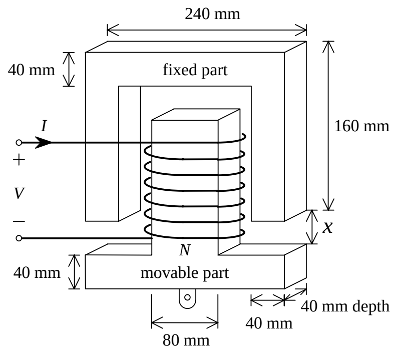




**2.**

The steel magnetic actuator shown below is excited by DC, with $I = 2\text{ A}$ and $N = 5000$ turns. The three gaps have the same length, *x*.

The mass density of the movable part is $7800\text{ kgm}^{-3}$.




The movable part is at a position for which $B_{steel} = 1.5\text{ T}$.

(a) Draw the magnetic equivalent circuit, assuming $\mu_{steel} = \infty$.

(b) Determine the air gap length, *x*, and the force exerted by the actuator.

(c) Determine the mass *m* (excluding actuator) which can be lifted against gravity.

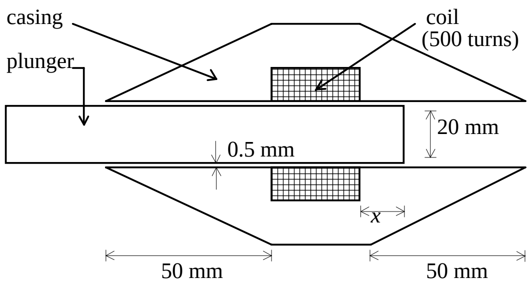
(d) Calculate the minimum current *I* needed to lift the unloaded actuator.

(e) Determine the initial acceleration of the unloaded actuator if it is released when $I = 0.3\text{ A}$.

**3.**

The magnetic actuator shown in the diagram has a cylindrical casing and a cylindrical plunger, both of the same magnetic material, which saturates at a flux density $B$ = 1.6 T. The plunger is free to move within the casing. The gap between the two is 0.5 mm. The diameter of the plunger is 20 mm.

The device is excited by a coil with $N = 500$ turns and carrying current $I$ and is capable of producing a small force over a large distance.




Determine the maximum coil current if $B$ is not to exceed 1.6 T.

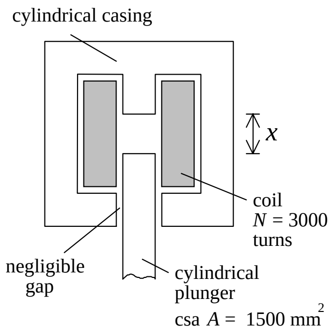
(Neglect leakage and fringing flux and assume the magnetic material to be perfect up to saturation).





**4.**

A magnetic actuator is constructed as shown:




The cylindrical plunger is free to move vertically. The coil resistance is $R = 8\,\Omega$. A 12 V DC supply is connected to the coil. The magnetic material is assumed perfect up to its saturation flux density of 1.6 T.

(a) Determine the $x$ range over which the force on the plunger is essentially constant because the material is saturated.

(b) Determine the mechanical energy produced when $x$ varies slowly from 10 mm to 0 mm.

(c) The plunger is allowed to close so quickly (from $x = 10\text{ mm}$) that the change in flux linkage is negligible. Calculate the mechanical energy produced.

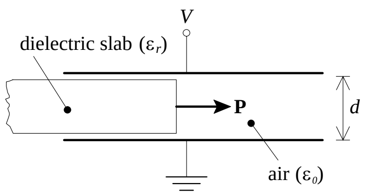
**5.**

A capacitor is formed by two parallel plates of csa $A = 0.02\text{ m}^2$, $x$ m apart. The dielectric is air. The plates are kept at a constant potential difference $V = 10\text{ kV}$. Derive an expression for the force $F$ between the plates as a function of $x$, and determine the energy converted to mechanical form as $x$ is reduced from 10 mm to 5 mm.

# 8B.12

**6.**

A rectangular block of dielectric material (relative permittivity $\varepsilon_r$) is partially inserted between two much larger parallel plates.



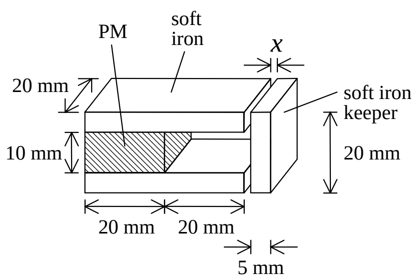

Show that the magnitude of the pressure on the RHS of the slab is:

$$P = \frac{\varepsilon_0(\varepsilon_r - 1)V^2}{2d^2}$$

Ignore fringing and leakage flux.

**7.**

A PM assembly to be used as a door holder was introduced in question 4B.5:




(a) Use the circuit derived in question 4B.5 (a) to obtain an expression for the force acting on the keeper as a function of $x$.

(b) Determine this force when $x$ = 1 mm.

# Lecture 9A – The Bipolar Junction Transistor

*Principle of the Bipolar Junction Transistor (BJT). Biasing circuit. Small signal equivalent circuit. Common emitter voltage amplifier. The emitter follower.*

## Principle of the Bipolar Junction Transistor (BJT)

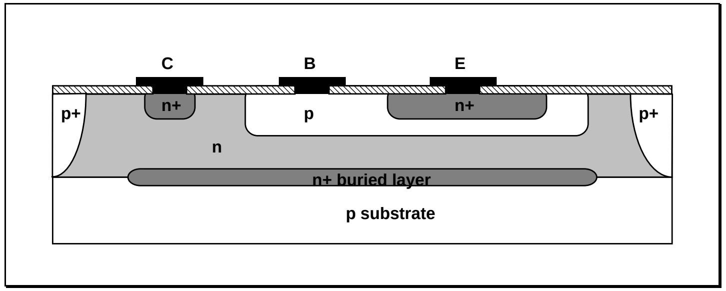



**Figure 9A.1**

A bipolar junction transistor is formed by the following process. Starting with a p-type substrate, a layer of heavily doped n-type material is deposited. This is the buried layer, and forms a low resistance path for what we will eventually call the collector current. An epitaxial layer of n-type semiconductor is deposited on top.

Isolation diffusion is then used to isolate the transistor. A base region is formed by very lightly diffusing p-type silicon into the structure. The emitter is formed by heavy n-type diffusion into the base. The result is an *npn* transistor.

We can simplify the above geometry to the following:

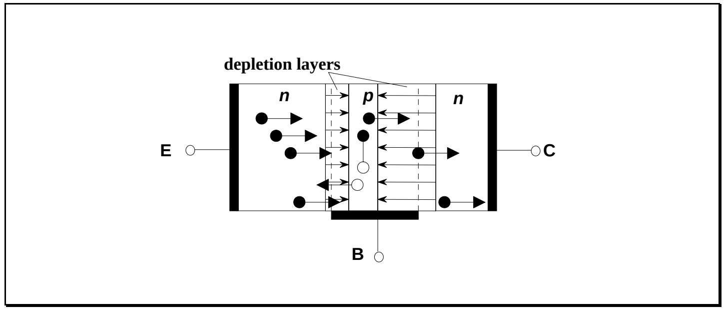


**Figure 9A.2**

The base-emitter junction (EBJ) is forward biased and the base-collector junction is reverse biased for "active mode" operation.

The forward bias on the EBJ causes a current across the junction. Electrons, which are the majority carriers in the emitter, are "injected" into the base. The base is lightly doped, so only some of the electrons will recombine with holes (the majority carriers in the base). The electrons in the base will be like gas molecules - they will diffuse in the direction of the concentration gradient - away from the EBJ. When they reach the collector-base depletion region, they are swept across by the field existing there (hence the electrons are collected).

The collector current is therefore seen to be independent of the CBJ reverse bias voltage. All that is necessary is a field of some magnitude between the collector and base.

There are two components to the base current. One is the holes injected from the base into the emitter. The other represents the holes that enter the base and recombine with electrons.

The charge carriers in the BJT are electrons and holes, hence the name bipolar junction transistor.

The collector current is almost equal to the current due to the forward biased EBJ, so it has an exponential relationship with $v_{BE}$. The base current, consisting almost entirely of injected holes, is much smaller, but still has an exponential relationship with $v_{BE}$. The collector current and base current are therefore proportional:

$$\begin{aligned} i_c &= \beta i_b \\ \beta &= \text{common-emitter current gain} \end{aligned} \tag{9A.1}$$

We can now look at the characteristics and explain their shape.





**Input and Output Characteristics**

The input characteristic is a graph of input current vs voltage, i.e. $I_B$ vs $V_{BE}$. It is equivalent to a forward biased *p-n* junction:

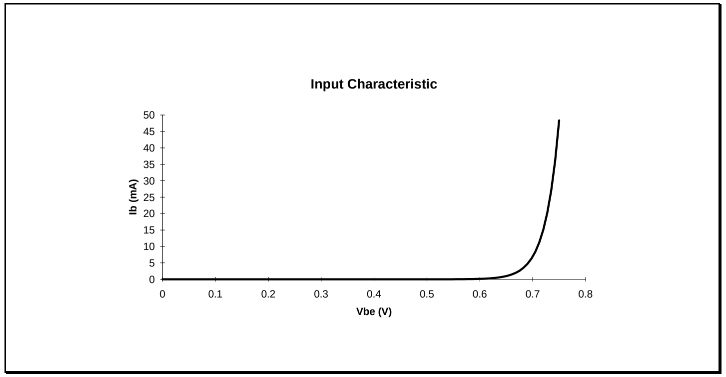



**Figure 9A.3**

The slope at a point on the characteristic is the inverse of the small signal input resistance of the device.

The output characteristic is a graph of $I_C$ vs $V_{CE}$ for particular values of $I_B$. It is similar to the MOSFET output characteristic, except now the output current is controlled by a current (instead of a voltage). By varying $I_B$ we obtain a family of characteristics:

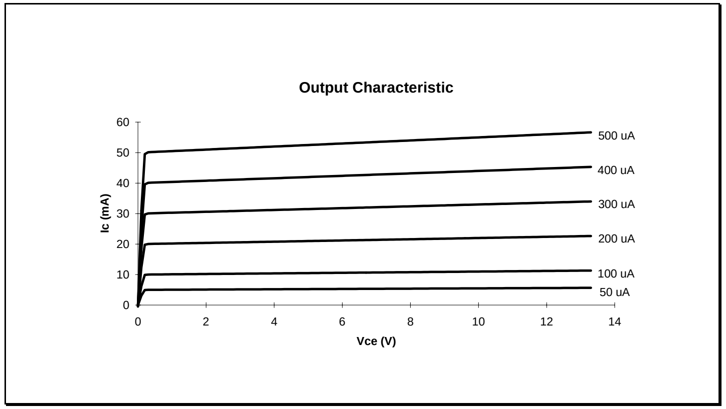



**Figure 9A.4**

In the active region, the collector current is nearly independent of the collector-emitter voltage. It is therefore like a current source. The collector current is set by the base current.

The slope of the curve in the active region is the inverse of the small signal output resistance of the device.

## Biasing Circuit

To use the BJT as an amplifier, we must ensure that it is in the active mode. The EBJ must be forward biased and the CBJ must be reverse biased. The biasing circuit should minimize the effect of transistor parameter variations. These may be caused by varying devices or changes in temperature.

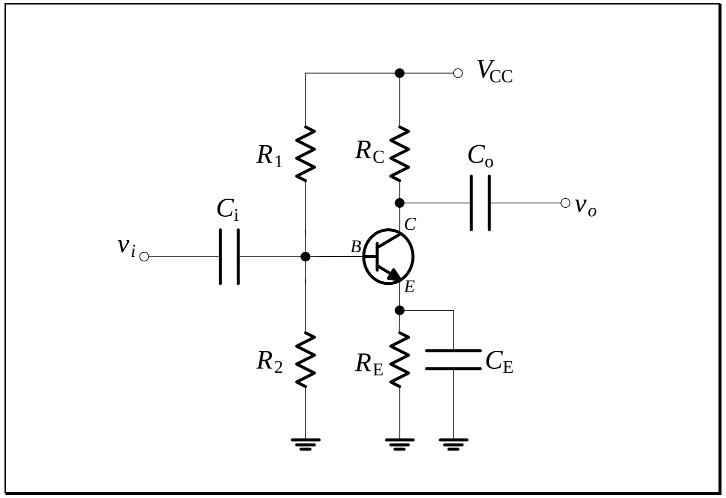



**Figure 9A.5**

Suppose we have chosen the $Q$ point ($I_{CQ}$ and $V_{CEQ}$ ). We now wish to design the bias circuit. The capacitors block all DC components of current and are equivalent to short circuits for AC. They are not considered in the DC analysis of the bias circuit.





The purpose of $R_E$ is to make the bias voltage $V_B = V_{BE} + V_E$ just large enough to be not affected by variations in $V_{BE}$ (due to temperature) and $r_{be}$ (due to device variations).

As $V_{BE} \approx 0.7$ V for silicon, we choose $V_E = 1$ V (no larger than 2 V). Therefore:

$$I_E = I_C + I_B \tag{9A.2a}$$

$$\approx I_{CQ} \tag{9A.2b}$$

$$R_E = \frac{V_E}{I_{CQ}} = \frac{1}{5} = 200\ \Omega \tag{9A.2c}$$

KVL from ground, through the BJT, to the DC supply gives:

$$V_E + V_{CEQ} + R_C I_C = V_{CC} \tag{9A.3}$$

We choose $I_C$ so that there can be equal positive and negative voltage excursions at the collector:

$$R_C I_C = V_{CEQ} \tag{9A.4}$$

$$R_C = \frac{V_{CEQ}}{I_{CQ}} = \frac{10}{5} = 2\ \text{k}\Omega$$

Therefore, the supply voltage will have to be $V_{CC} = 10 + 10 + 1 = 21$ V.

We would like to keep the input resistance to the amplifier large, so we want to choose the resistors $R_1$ and $R_2$ as large as possible. However, to keep the voltage at the base independent of the transistor parameters (the base current $I_B$ will vary between devices for a given $I_C$), we want the current in the potential divider to be larger than the base current.

A trade off is necessary, so we choose the current through $R_1$ and $R_2$ to be $0.1I_{CQ}$:

$$R_1 + R_2 = \frac{V_{CC}}{0.1I_{CQ}} = \frac{21}{0.5} = 42\ \text{k}\Omega \tag{9A.5}$$

To obtain the individual values, we apply KVL around the base-emitter loop:

$$R_2\, 0.1I_{CQ} - V_{BE} - V_E = 0$$

$$R_2 = \frac{V_{BE} + V_E}{0.1I_{CQ}} = \frac{1.7}{0.5} = 3.4\ \text{k}\Omega \tag{9A.6a}$$

$$R_1 = 42 - 3.4 = 38.6\ \text{k}\Omega \tag{9A.6b}$$





### Small Signal Equivalent Circuit

From the output characteristic, we can see that the output current is controlled by $i_b$ and $v_{ce}$:

$$i_c = \beta i_b + y_{ce} v_{ce} \tag{9A.7}$$

This is valid for mid frequencies, where we can ignore the *p*-*n* junction capacitances.

The small signal equivalent circuit is therefore:

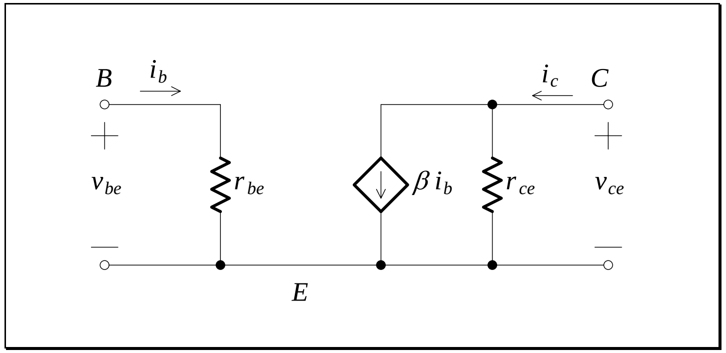



**Figure 9A.6**

Notice how the dependent source in this model is current controlled. We can make it voltage controlled by saying:

$$\beta i_b = g_m v_{be} = g_m r_{be} i_b$$

$$\therefore g_m = \frac{\beta}{r_{be}} \tag{9A.8}$$

## Common Emitter Voltage Amplifier

For small AC signals, the BJT small signal equivalent circuit can be used in the AC analysis. We note also that the capacitor $C_E$ by-passes (effectively shorts) the resistor $R_E$. The DC supply is equivalent to a short circuit so that resistor $R_C$ is connected to common. The input and output capacitors $C_i$ and $C_o$ are assumed to have zero reactance. It is very similar to the MOSFET common source voltage amplifier.

The small signal AC equivalent circuit becomes:

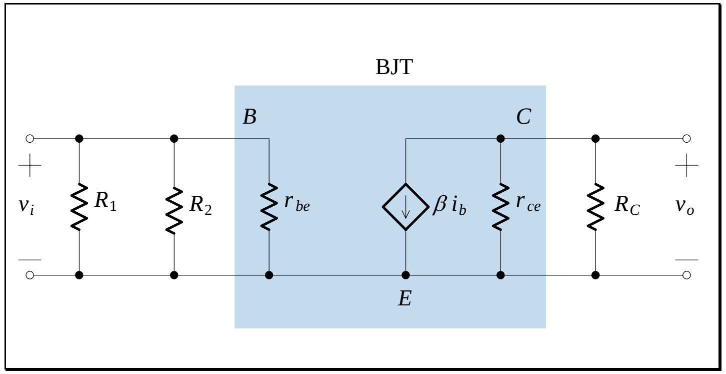



**Figure 9A.7**

The parameters of the voltage amplifier we need for our "amplifier block" representation are:

$$A_{vo} = \frac{v_o}{v_i} = -\frac{\beta i_b (r_{ce} \parallel R_C)}{i_b r_{be}}$$

$$\approx -\frac{\beta}{r_{be}} R_C = -g_m R_C \tag{9A.9a}$$

$$R_{\text{in}} = R_1 \parallel R_2 \parallel r_{be} \tag{9A.9b}$$

$$R_{\text{out}} = r_{ce} \parallel R_C \approx R_C \tag{9A.9c}$$





## The Emitter Follower

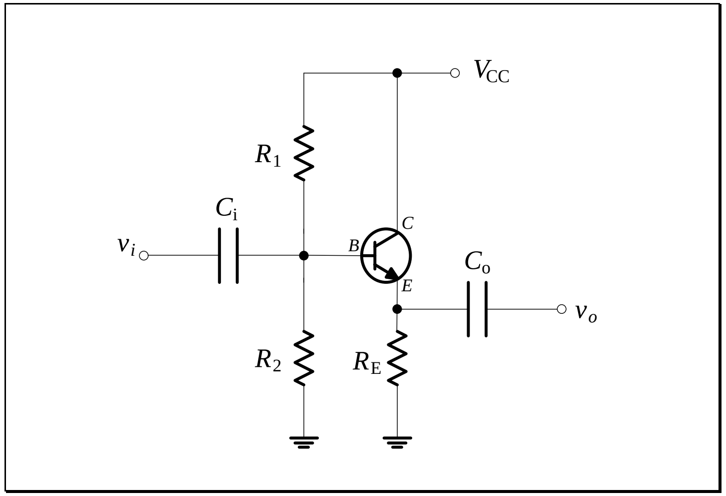



**Figure 9A.8**

Here, the output is taken from the emitter terminal, and the collector, for AC is connected to the common.

The small signal equivalent circuit for the above amplifier is:

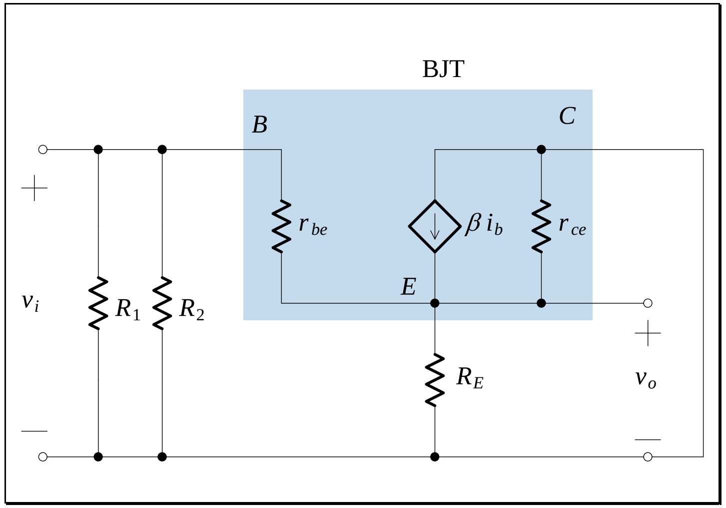



**Figure 9A.9**

KCL at the emitter gives:

$$\beta i_b + i_b = \frac{v_o}{r_{ce} \parallel R_E}$$

$$(\beta+1)\frac{v_i - v_o}{r_{be}} = \frac{v_o}{r_{ce} \parallel R_E}$$

$$\frac{v_i}{r_{be}/(\beta+1)} = \left(\frac{1}{r_{ce} \parallel R_E} + \frac{1}{r_{be}/(\beta+1)}\right)v_o$$

$$\frac{v_o}{v_i} = \frac{1}{\frac{r_{be}/(\beta+1)}{r_{ce} \parallel R_E} + 1}$$

$$A_{vo} = \frac{r_{ce} \parallel R_E}{r_{ce} \parallel R_E + r_{be}/(\beta+1)} \approx 1 \tag{9A.10}$$

KVL at the base gives:

$$v_b = r_{be} i_b + (r_{ce} \parallel R_E)(\beta+1) i_b$$

$$R_{\text{in}} = \frac{v_b}{i_b} = r_{be} + (r_{ce} \parallel R_E)(\beta+1) = \text{large} \tag{9A.11}$$

We can first find the short circuit current, and using the open circuit voltage, find the output resistance:

$$i_{sc} = (\beta+1) i_b = (\beta+1)\frac{v_i}{r_{be}}$$

$$\begin{aligned} R_{\text{out}} &= \frac{v_{oc}}{i_{sc}} = \frac{r_{ce} \parallel R_E}{r_{ce} \parallel R_E + r_{be}/(\beta+1)} \frac{r_{be}}{(\beta+1)} \\ &= r_{be}/(\beta+1) \parallel r_{ce} \parallel R_E = \text{small} \end{aligned} \tag{9A.12}$$





**References**

Sedra, A. and Smith, K.: *Microelectronic Circuits*, Saunders College Publishing, New York, 1991.

# Lecture 9B – The Moving Coil Machine

*Generator principle. Motor (or meter ) principle.*

## Generator Principle

A simple generator consists of a rectangular coil ($N_1$ turns, radius $r_1$, area $A_1$) wound on a soft iron cylindrical core (the rotor). The pole faces are shaped so that the air gap is uniform and the gap flux density, $B_g$, is perpendicular to the pole faces. The rotor (and therefore the coil) is driven at constant angular speed $\omega_1$ by a "prime mover" – such as a steam turbine or a diesel motor.

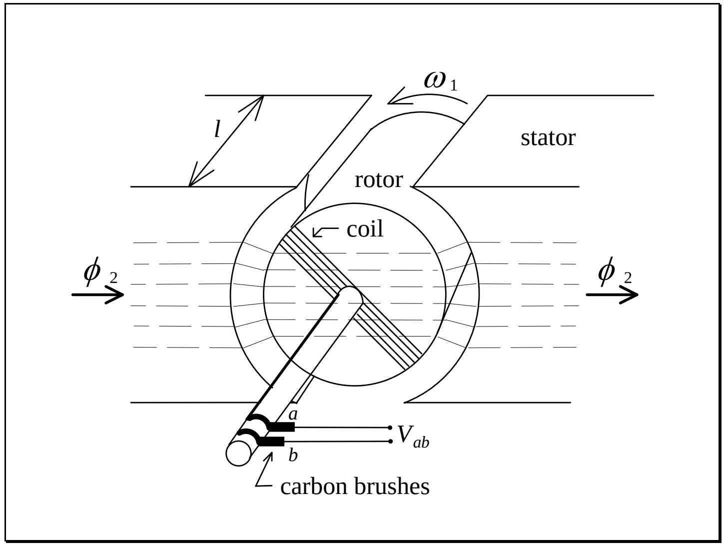



**Figure 9B.1**

The coil rotation is equivalent to a conductor of length $l$ moving with velocity **v** in a magnetic field of density $B_g$, as shown below:

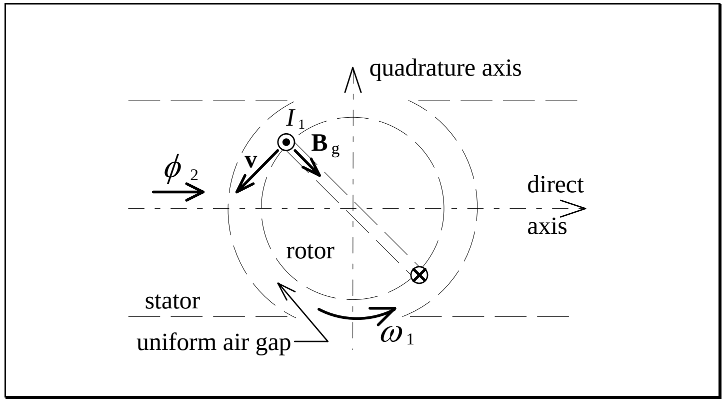



**Figure 9B.2**

We can apply the Lorentz force to a free charge in the conductor, and then calculate the potential difference between the brushes:

$$\mathbf{E} = -(\mathbf{v} \times \mathbf{B}), \qquad V_{ab} = -\int_b^a \mathbf{E} \cdot d\mathbf{l}$$

$$\begin{aligned} V_{ab} &= N_1 \int_a^b (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} \\ &= N_1 \omega_1 r_1 B_g 2l \\ &= B_g A_1 N_1 \omega_1 \end{aligned} \tag{9B.1}$$

A pulsating emf $V_{ab}$ results. A DC output can be obtained by connecting leads $a$ and $b$ to a commutator which switches the output "positive" terminal to the other slip ring when the polarity changes.

To create the magnetic field, we can use permanent magnets or a coil.





Assuming we have a good core, the mmf required to produce the magnetic field in the air gap is given by:

$$N_2 I_2 \approx \mathcal{R}_g \phi_g = \frac{2l_g}{\mu_0 A_g} B_g A_g = KB_g \tag{9B.2}$$

The mmf does not depend upon the radius of the machine. It depends upon the length.

The electrical power output from the generator is limited by factors such as:

- Overheating of windings. To compensate for overheating, windings are distributed over the periphery of the rotor for better heat dissipation.
- The number of conductors that can be accommodated (and therefore the current) is proportional to the radius.
- The output voltage is limited by saturation in the iron ($\phi_2$ controls $B_g$).

The output power of the machine is given by:

$$VI = KA_1 \omega_1 r_1 = K' \omega_1 \times (\text{machine volume}) \tag{9B.3}$$

Power station generators therefore tend to be large. With these large machines, the large currents are unsuitable for brush contacts. In these machines, the field and rotor windings are reversed so that the brushes carry the field current. The field supply is either by battery or rectified AC.

**Advantage of AC Generation**

Alternating current and voltage are easily transformed (the transformer is simple and very efficient – 98%).

**Generation of Sinusoidal Voltages**

The pole faces are shaped to give a sinusoidal $B_g$ or the windings are distributed so that the mmf is sinusoidal.

### Electrical Equivalent Circuit

The electrical equivalent circuit for a machine is:

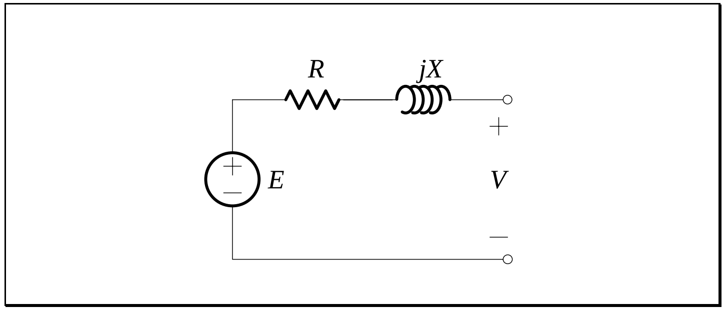



**Figure 9B.3**

## Motor (or Meter) Principle

To use the machine as a motor, we apply a voltage $V_{ab}$ to terminals *a* and *b*. There is a current $I_1$ as shown:

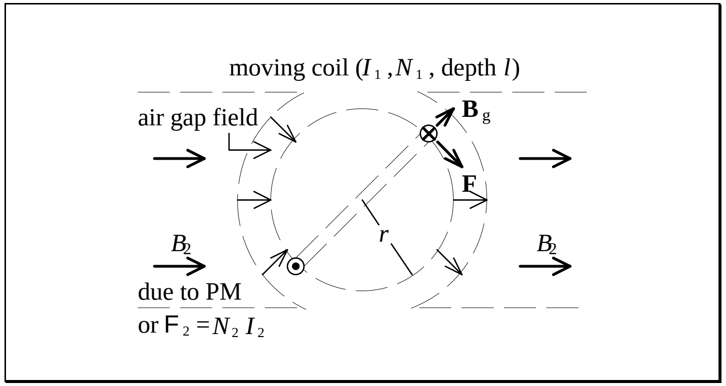



**Figure 9B.4**





The force and torque experienced by an element *dl* of coil 1 (the moving coil) is:

$$d\mathbf{F} = I_1\left(d\mathbf{l} \times \mathbf{B}_g\right) = I_1 dl B_g$$

$$T = r\int_b^a dF = B_g A_1 N_1 I_1 \tag{9B.4}$$

The moving coil moves clockwise until $T = 0$ is reached. To operate the machine as a motor, an AC source is needed to produce continuous motion – in a DC machine the voltage is switched via the commutator on the rotor.

### Moving Coil Meter

The moving coil is free to rotate in the air gap field. The magnetic field is produced by a permanent magnet. The current to be measured ($I_1$) is fed via the control springs:

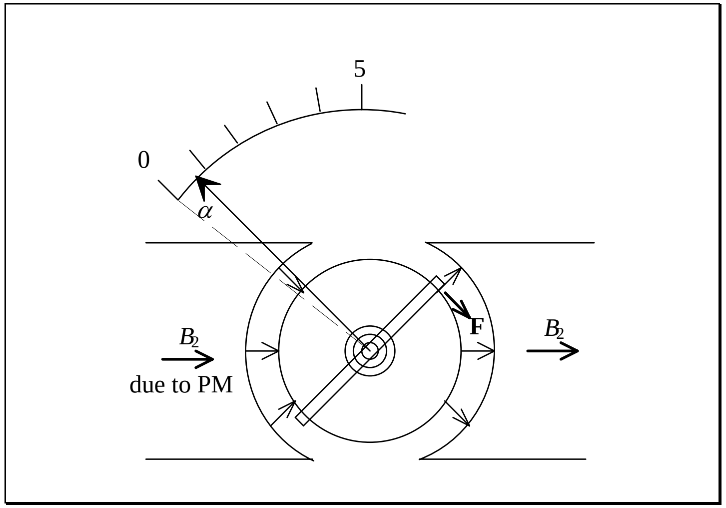



**Figure 9B.5**

The electric deflecting torque $T_d$ (due to $I_1$ and $B_g$) causes rotation of the moving coil (and the attached pointer) against the restoring torque of the spring ($T_r$).

If $B_g$ is constant, then:

$$T_d = K_d I_1 \qquad K_d = B_g A_1 N_1$$
$$T_r = K_r \alpha \qquad K_r = \text{spring constant} \tag{9B.5}$$

At balance:

$$T_d = T_r \tag{9B.6}$$

and the deflection of the pointer is:

$$\alpha = K I_1 \tag{9B.7}$$

The scale of a moving coil meter is therefore linear (uniformly divided).

For time varying currents, the pointer will respond to the average deflecting torque (due to the inertia of the coil):

$$T_{d_{AV}} = \frac{1}{T}\int_0^T T_d dt = K_d \frac{1}{T}\int_0^T i_1 dt = K_d I_{1_{AV}}$$
$$\alpha = K I_{1_{AV}} \tag{9B.8}$$

If we apply a sinusoidal current, then:

$$i_1 = \hat{I}_1 \cos \omega t$$
$$I_{1_{AV}} = 0 \tag{9B.9}$$





To obtain a reading, $i_1$ is rectified.

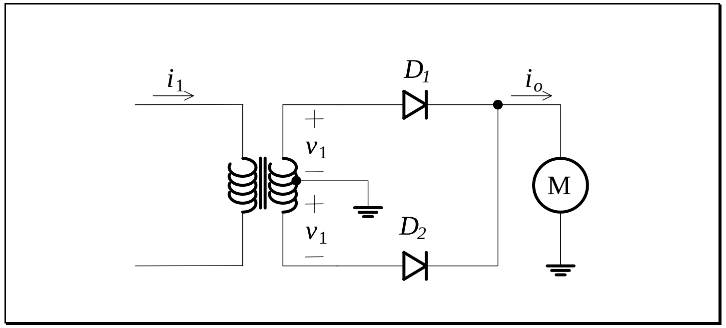



**Figure 9B.6**

For a full wave rectified (FWR) sine wave:

$$I_{AV} = \frac{2}{\pi}\hat{I}$$
$$I_{RMS} = \frac{\hat{I}}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}} I_{AV} = 1.11 I_{AV} \tag{9B.10}$$

The scale may therefore be calibrated in RMS values (average x 1.11).

The moving coil meter responds to the average value, so AC moving coil meters use a FWR to read sine wave RMS values.

## References

Slemon, G. and Straughen, A.: *Electric Machines*, Addison-Wesley Publishing Company, Inc., Sydney, 1982.

# Lecture 10A - Frequency Response

*The amplifier block. Voltage and current amplifiers. Maximum power transfer. The decibel (dB). Frequency response of capacitively coupled circuits.*

## The Amplifier Block

The amplifier is a basic building block of analog electronic systems. Its response may be modelled without a knowledge of what is inside, which may change with technical innovation.

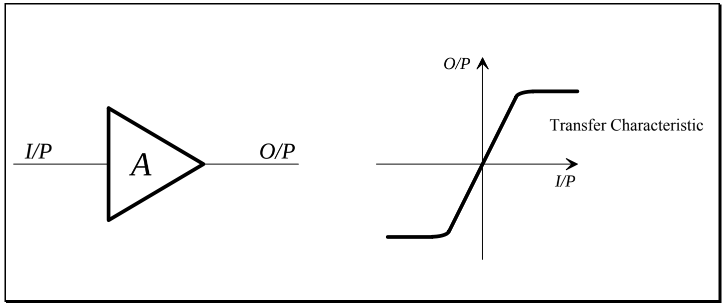



**Figure 10A.1**

If we treat an amplifier as a block, what do we have to know about it? The important parameters of an amplifier are:

- the open circuit gain $A$
- the input impedance $Z_i$
- the output impedance $Z_o$

To analyze the suitability of an amplifier in an application, we need to know the external circuit.

## Voltage Amplifier

A voltage amplifier is used to amplify a voltage:

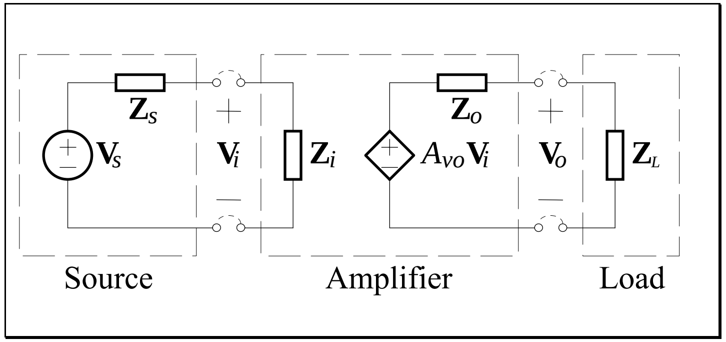



**Figure 10A.2**

where:

- $A_{vo}$ = open circuit voltage gain
- $\mathbf{Z}_i$ = large
- $\mathbf{Z}_o$ = small

because:

$$\mathbf{V}_i = \frac{\mathbf{Z}_i}{\mathbf{Z}_i + \mathbf{Z}_s}\mathbf{V}_s \tag{10A.1a}$$

$$\mathbf{V}_L = \mathbf{V}_o = \frac{\mathbf{Z}_L}{\mathbf{Z}_L + \mathbf{Z}_o}A_{vo}\mathbf{V}_i \tag{10A.1b}$$





## Current Amplifier

A current amplifier is used to amplify a current:

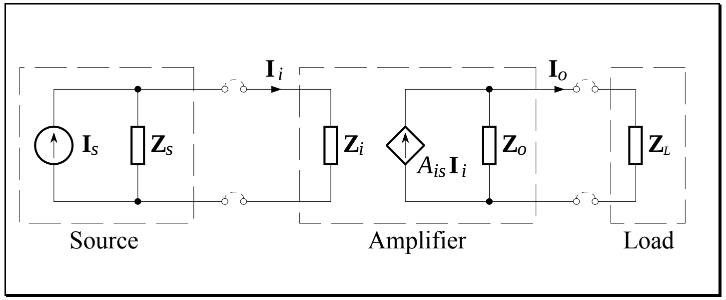



**Figure 10A.3**

where:

- $A_{is}$ = short circuit current gain
- $\mathbf{Z}_i$ = small
- $\mathbf{Z}_o$ = large

because:

$$\mathbf{I}_i = \frac{\mathbf{Z}_s}{\mathbf{Z}_s + \mathbf{Z}_i}\mathbf{I}_s \tag{10A.2a}$$

$$\mathbf{I}_L = \mathbf{I}_o = \frac{\mathbf{Z}_o}{\mathbf{Z}_o + \mathbf{Z}_L}A_{is}\mathbf{I}_i \tag{10A.2b}$$

## Maximum Power Transfer

Consider the following circuit:

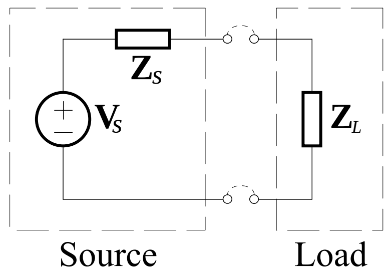



**Figure 10A.4**

To achieve maximum power in the load, what impedance do we choose? The load power is given by:

$$P_L = |\mathbf{I}_L|^2 R_L = \frac{|\mathbf{V}_s|^2}{|\mathbf{Z}_L + \mathbf{Z}_s|^2} R_L$$

$$= \frac{|\mathbf{V}_s|^2 R_L}{(R_L + R_s)^2 + (X_L + X_s)^2} \qquad (10A.3)$$

To make the power as large as possible, make $(X_L + X_s)^2$ zero by choosing $X_L = -X_s$. We have seen before that to maximize power with just resistors, we choose $R_L = R_s$. In order to get maximum power to $Z_L$ we select:

$$\mathbf{Z}_L = R_L + jX_L = R_s - jX_s = \mathbf{Z}_s^* \qquad (10A.4)$$





The maximum power is therefore:

$$P_{L(\max)} = \frac{|\mathbf{V}_s|^2}{4R_L} \qquad (10A.5)$$

Power matching is used in three situations:

- where the signal levels are very small, so any power lost gives a worse signal to noise ratio. e.g. in antenna to receiver connections in television, radio and radar.
- high frequency electronics
- where the signal levels are very large, where the maximum efficiency is desirable on economic grounds. e.g. a broadcast antenna.

## The Decibel (dB)

Historically the Bel (named after Alexander Graham Bell – the inventor of the telephone) was used to define ratios of audio loudness i.e. ratios of power:

$$\text{Power gain (B)} = \log_{10} \frac{P_o}{P_i} \qquad (10A.6)$$

Through the use of the metric system, a convenient unit to use was the decibel (dB):

$$\text{Power gain (dB)} = 10\log_{10} \frac{P_o}{P_i} \qquad (10A.7)$$

The previous equation can also express the relationship between output power and input power for an amplifier. If we know what resistances the input and output power are dissipated in, and the voltages across them, then we can write:

$$\begin{aligned}\text{Power gain (dB)} &= 10\log_{10}\frac{|\mathbf{V}_o|^2 / R_0}{|\mathbf{V}_i|^2 / R_i} \\ &= 20\log_{10}\left|\frac{\mathbf{V}_o}{\mathbf{V}_i}\right| + 10\log_{10}\frac{R_i}{R_o} \qquad \text{(10A.8)}\end{aligned}$$

In any circuits where $R_i = R_o$, then, and *only* then:

$$\begin{aligned}\text{Power gain (dB)} &= 20\log_{10}\left|\frac{\mathbf{V}_o}{\mathbf{V}_i}\right| \\ &= 20\log_{10}(\text{voltage gain}) \qquad \text{(10A.9)}\end{aligned}$$

In many systems (e.g. communication systems), this relationship was applicable, so eventually it became customary to express *voltage gain* in terms of the decibel.

The dB unit of power gain is useful when circuits are cascaded – you can add the voltage gains in dB instead of multiplying the standard voltage gains.

The frequency response curves of circuits are simple when the gain in decibels is plotted against frequency on a logarithmic scale, and again they are easily added when circuits are connected in cascade.





### Frequency Response of Capacitively Coupled Circuits

The transistor amplifier looked at so far contains capacitors to couple the source and load to the amplifier. The amplifier performance is therefore dependent upon the frequency of the signal source. If we examine what happens to one of the amplifier's parameters when the frequency is varied, then we are looking at the frequency response of the parameter. The parameter we are most interested in at this stage is the gain of the amplifier.

Consider the output circuit of a voltage amplifier:

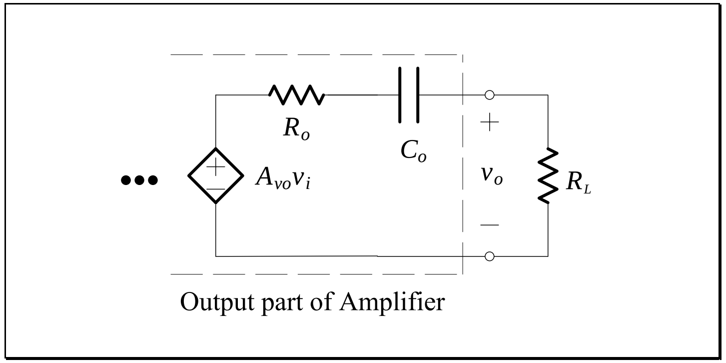



**Figure 10A.5**

The expression for the gain (as a function of frequency) can be derived:

$$\begin{aligned}\mathbf{V}_o = R_L\mathbf{I}_L &= R_L\frac{A_{vo}\mathbf{V}_i}{(R_o + R_L) + 1/j\omega C_o} \\ \therefore \text{Gain } \mathbf{A}_v = \frac{\mathbf{V}_o}{\mathbf{V}_i} &= A_{vo}\frac{j\omega C_o R_L}{1 + j\omega C_o (R_o + R_L)} \qquad \text{(10A.10)}\end{aligned}$$

Lets see how the expression for gain changes with the frequency. We will consider three cases.

**Mid-Frequencies**

The mid-frequency is chosen by the amplifier designer, and the values of the capacitors are chosen to suit. At mid-frequencies, we assume the capacitor has been chosen so that its reactance is very small compared to the resistors.

$$\left(R_o + R_L\right) >> \frac{1}{\omega C_o} \tag{10A.11a}$$

$$\text{Mid-frequency gain } A_{vM} = A_{vo}\frac{R_L}{R_o + R_L} \tag{10A.11b}$$

**Corner Frequency**

At a frequency $\omega_o$ such that:

$$\left(R_o + R_L\right) = \frac{1}{\omega_o C_o} \tag{10A.12}$$

the gain is:

$$\begin{aligned} \text{Gain } \mathbf{A}_v &= A_{vo}\frac{R_L}{\left(R_o + R_L\right) + \left(R_o + R_L\right)/j} \\ &= A_{vo}\frac{R_L}{\left(R_o + R_L\right)}\frac{1}{1 - j} \\ &= \frac{A_{vM}\left(1 + j\right)}{2} = \frac{A_{vM}}{\sqrt{2}}\angle 45° \end{aligned} \tag{10A.13}$$

The gain is 70.7% of the mid-frequency gain, and the output leads the input by 45°.





**Low Frequencies**

The third region of interest is at very low frequencies, well below the corner frequency. At these frequencies:

$$\left(R_o + R_L\right) << \frac{1}{\omega C_o}$$

$$\begin{aligned} \text{Gain } \mathbf{A}_v &= A_{vo}\frac{R_L}{1/j\omega C_o} = jA_{vo}\omega R_L C_o \\ &= jA_{vM}\omega C_o\left(R_o + R_L\right) \\ &= jA_{vM}\frac{\omega}{\omega_o} = A_{vM}\frac{\omega}{\omega_o}\angle 90° \end{aligned} \tag{10A.14}$$

At very low frequencies, the gain is proportional to the frequency. At DC the gain is zero, as expected.

At high frequencies, the frequency response of the whole amplifier must be considered. The input circuit, stray "feedback" capacitance, and capacitance internal to any transistors would have to be considered. The overall effect is a reduction in the gain, making it difficuolt to design amplifiers for high frequency operation.

### Graphical Analysis

For graphical analysis, we plot the magnitude of the gain versus frequency and the phase of the gain versus frequency. These two plots together constitute a Bode* plot. We normally use logarithmic scales for each axis. For example:

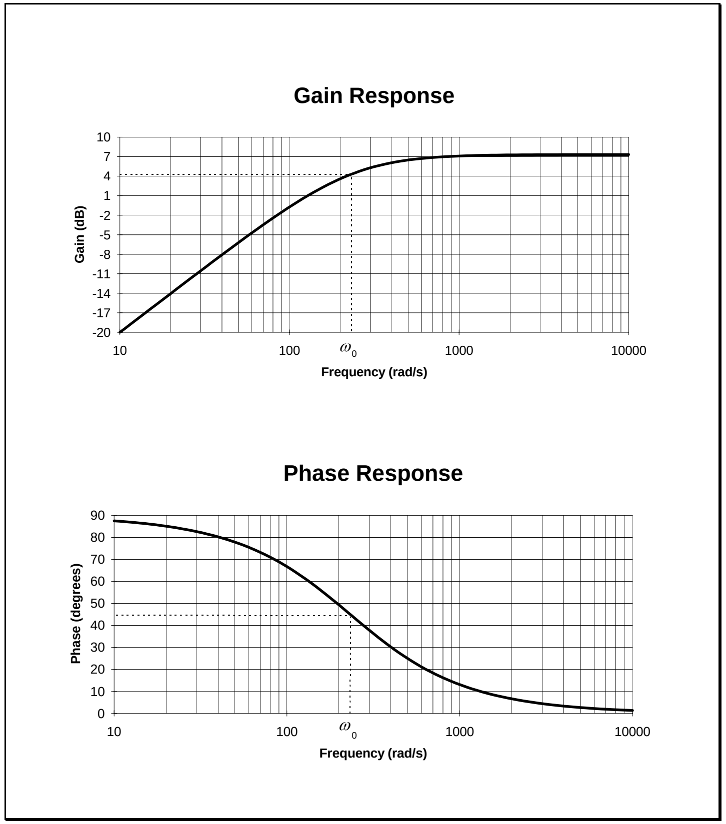



**Figure 10A.6**

* Dr. Hendrik Bode grew up in Urbana, Illinois, USA, where his name is pronounced *boh-dee*. Purists insist on the original Dutch *boh-dah*. No one uses *bohd*.





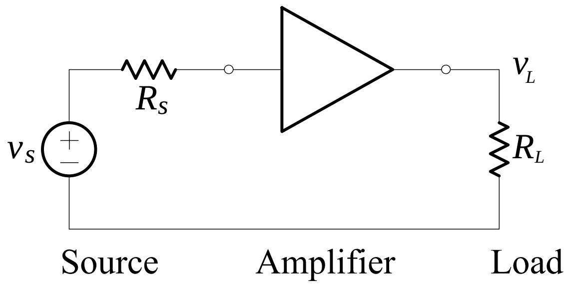
## References

Sedra, A. and Smith, K.: *Microelectronic Circuits*, Saunders College Publishing, New York, 1991.

## Problems

**1.**

The following circuit is used to perform tests on a voltage amplifier:




The following results were obtained:

| $v_S$ (volts RMS) | $R_S$ (kΩ) | $v_L$ (volts RMS) | $R_L$ (kΩ) |
|---|---|---|---|
| 0.1 | 0 | 8 | ∞ |
| 0.1 | 0 | 4 | 1 |
| 0.1 | 1000 | 2 | 1 |

Determine the amplifier's input and output resistance and its open-circuit voltage gain $A_{vo}$.

Draw the equivalent circuit.

**2.**

A MOSFET is used in a single-stage common-source voltage amplifier that has a gain of 40 with a load resistor of $40\,\text{k}\Omega$. If the load resistance is halved, the voltage gain drops to 30.

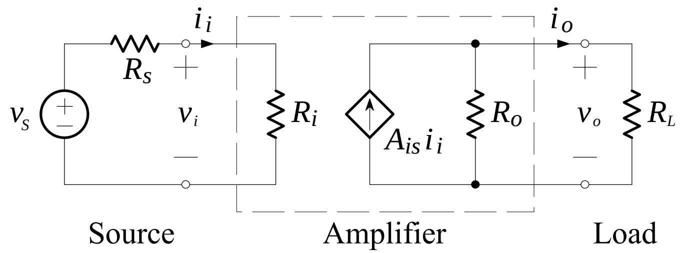
Determine the output resistance and the mutual conductance of the transistor.





**3.**

The diagram shows the Norton equivalent circuit of an amplifier:



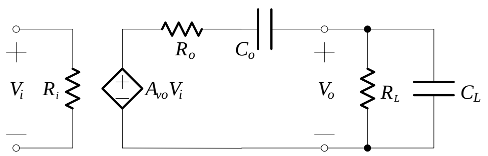

Derive an expression for the output current $i_o$ in terms of $A_{is}i_i$, $R_o$ and $R_L$.

If $i_o$ is to be within ±1% of $A_{is}i_i$, what is the relation between $R_o$ and $R_L$?

Determine $A_{is}i_i$ in terms of $A_{vo}$, $v_i$ and $R_o$.

**4.**

For the following circuit:




$A_{vo} = 100$
$R_o = 1\,\text{k}\Omega \quad C_o = 16\,\mu\text{F}$
$R_L = 9\,\text{k}\Omega \quad C_L = 180\,\text{pF}$

Determine the magnitude of the gain, $A_v = V_o/V_i$ at 1 Hz, 1 kHz and 1 MHz.

What is the amplifier's bandwidth between half-power points?

**5.**

A MOSFET is used in a single-stage common-source voltage amplifier with a load resistance of $45\,\text{k}\Omega$. When the load resistor is halved, the voltage gain reduces to 91% of its original value.

Determine the voltage gain of the circuit, for both loads, if $g_m = 6\,\text{mS}$.

# Lecture 10B – Bridges and Measurements

*General bridge equations. Measurement of resistance, inductance and capacitance. Average and RMS values of periodic waveforms.*

## General Bridge Equations

The general bridge is constructed as follows:

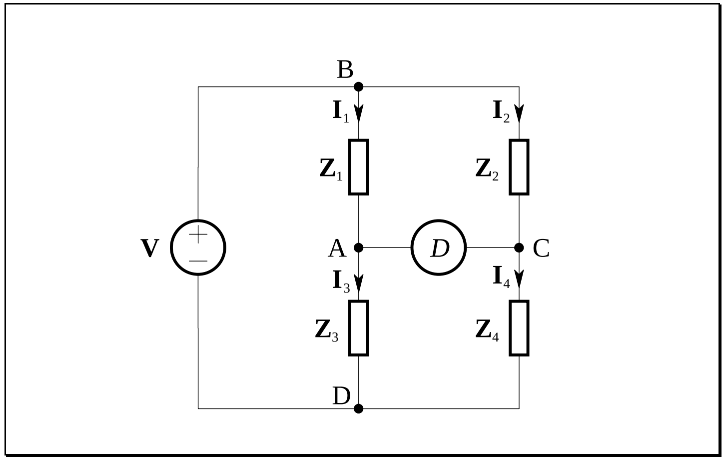



**Figure 10B.1**

The circuit is easy to analyze once there is no deflection in the detector *D*. When the circuit is in this state, it is said to be balanced, and:

$$\mathbf{V}_{AD} = \mathbf{V}_{CD}, \quad \mathbf{I}_1 = \mathbf{I}_3, \quad \mathbf{I}_2 = \mathbf{I}_4 \tag{10B.1}$$

Therefore:

$$\mathbf{Z}_1\mathbf{I}_1 = \mathbf{Z}_2\mathbf{I}_2 \quad \text{and} \quad \mathbf{Z}_3\mathbf{I}_1 = \mathbf{Z}_4\mathbf{I}_2 \tag{10B.2}$$

giving:

$$\frac{\mathbf{Z}_1}{\mathbf{Z}_2} = \frac{\mathbf{Z}_3}{\mathbf{Z}_4} \tag{10B.3}$$

## Measurement of Resistance

### Wheatstone bridge

The Wheatstone bridge (invented by Samuel Hunter Christie but popularized by Charles Wheatstone) is the simplest of all bridges, and is the most widely used method for the precision measurement of resistance. The Wheatstone bridge consists of four resistance arms, together with a source of current (a battery) and a detector (a galvanometer). The circuit is shown below:

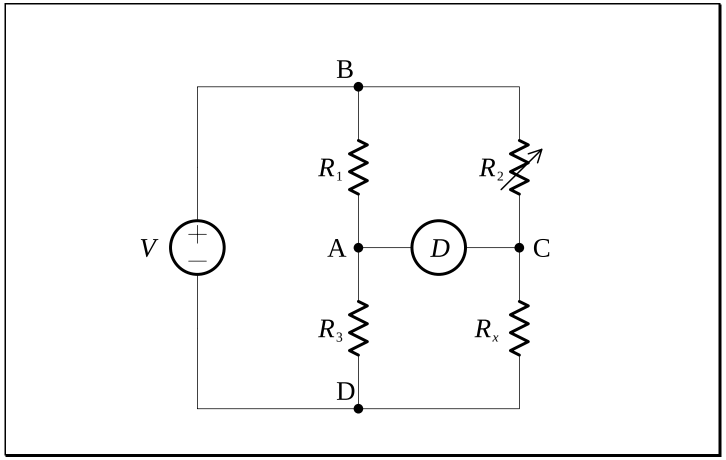



**Figure 10B.2**

It is used to measure resistance from 1 Ω to 10 MΩ. It has a low sensitivity for low values of resistance.

All the impedances of the general bridge are resistors, with $R_2$ being variable. $R_1$ and $R_3$ are at a fixed ratio. At balance:

$$R_x = \frac{R_3}{R_1} R_2 \tag{10B.4}$$





### Kelvin bridge

The Kelvin bridge may be regarded as a modification of the Wheatstone bridge to secure increased accuracy in the measurement of low resistance. It is used to measure resistance from 100 nΩ to 1 Ω.

An understanding of the Kelvin arrangement may be obtained by a study of the difficulties that arise in a Wheatstone bridge in the measurement of resistances that are low enough for the resistance of leads and contacts to be appreciable in comparison. Consider the bridge shown below, where $P$ represents the resistance of the lead that connects from $R_2$ to $R_x$:

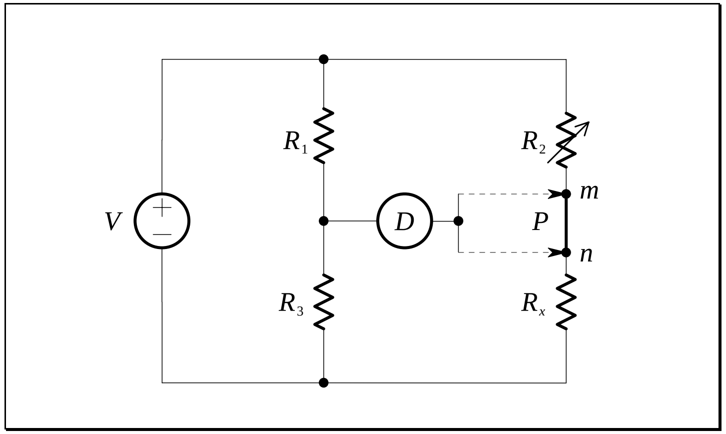



**Figure 10B.3**

Two possible connections for the galvanometer are indicated by the dotted lines. With connection to $m$, $P$ is added to $R_x$, so that the computed value of the unknown is higher than $R_x$ alone, if $P$ is appreciable in comparison with $R_x$. On the other hand, if connection is made to $n$, $R_x$ is in fact computed from the known value of $R_2$ only, and is accordingly lower than it should be.

Suppose that instead of using point $m$, which gives a *high* result, or $n$, which makes the result *low*, we can slide the galvanometer connection along to any desired intermediate point, as shown below:

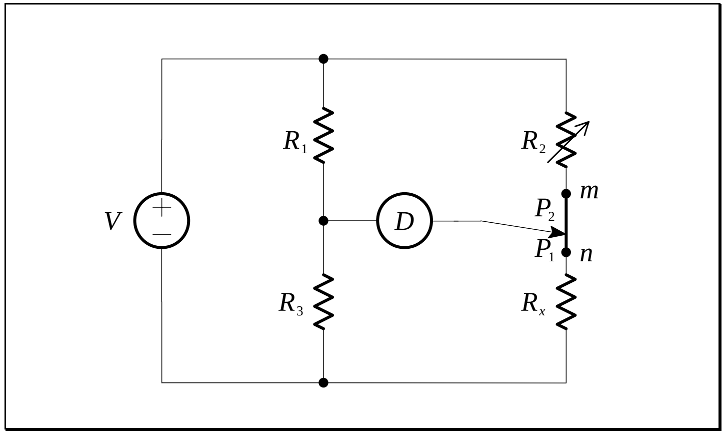



**Figure 10B.4**

From the usual balance relationship, we can write:

$$R_x + P_1 = \frac{R_3}{R_1}\left(R_2 + P_2\right) \tag{10B.5}$$

$$R_x = \frac{R_3}{R_1}R_2 + \frac{R_3}{R_1}P_2 - P_1$$

If the resistance of $P$ is divided into two parts such that

$$\frac{P_1}{P_2} = \frac{R_3}{R_1} \tag{10B.6}$$





then the presence of $P$ *causes no error* in the result, since substituting Eq. (10B.6) into Eq. (10B.5) gives:

$$R_x = \frac{R_3}{R_1}R_2 \tag{10B.7}$$

The final balanced condition has the same formula as the Wheatstone bridge.

The process described here is obviously not a practical way of achieving the desired result, as we would have trouble in determining the correct point for the galvanometer connection. It does, however, suggest the simple modification that we connect two actual resistance units of the correct ratio between points $m$ and $n$, and connect the galvanometer to their junction. This is the Kelvin bridge arrangement, shown below:

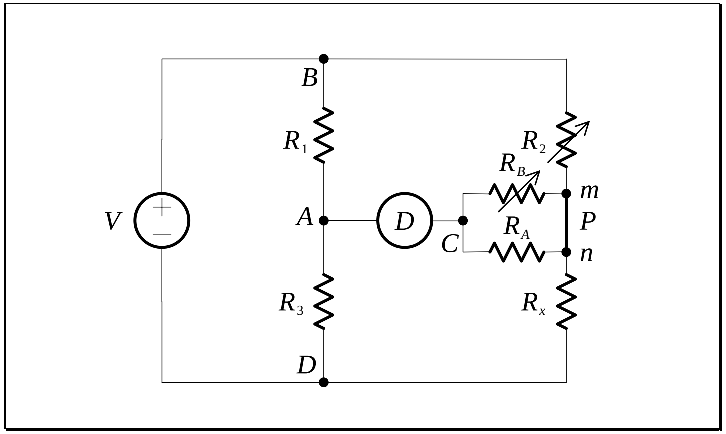



**Figure 10B.5**

To operate the bridge, a balance is performed in the normal way. Then the link $P$ is removed to see whether the detector still indicates balance. If it is out of balance then resistor $R_B$ is varied to balance the resulting Wheatstone bridge ($R_2$ and $R_x$ are negligible in comparison to $R_A$ and $R_B$ in this case).

With the link $P$ in place, the balance condition gives $V_{BA} = V_{BmC}$. The voltage $V_{BA}$ is given by:

$$V_{BA} = \frac{R_1}{R_1 + R_3} V$$
$$= \frac{R_1}{R_1 + R_3} I_2 \left( R_2 + R_x + \frac{(R_A + R_B)P}{R_A + R_B + P} \right) \quad (10B.8)$$

since no current exists in the galvanometer branch. Similarly, using the current divider rule, we get:

$$V_{BmC} = I_2 \left( R_2 + R_B \frac{P}{R_A + R_B + P} \right) \quad (10B.9)$$

If these two values are equated and $R_x$ made the subject, the result is:

$$R_x = \frac{R_3}{R_1} R_2 + \frac{R_B P}{R_A + R_B + P} \left( \frac{R_3}{R_1} - \frac{R_A}{R_B} \right) \quad (10B.10)$$

Now, if $R_3/R_1 = R_A/R_B$, the equation becomes:

$$R_x = \frac{R_3}{R_1} R_2 \quad (10B.11)$$

This indicates that the resistance of the connection $P$ (which carries most of the current) has no effect, provided that the two sets of ratio arms have equal ratios.





**Four-Terminal Resistor**

One complication in the construction of the Kelvin bridge is the fact that in actual practice the resistance $P$ includes not only the ohmic resistance of the connecting wire, but also the contact resistance between wire and binding post. Contact resistance is a variable and uncertain element, as it depends on such things as the cleanness of the surfaces and the amount of pressure between them. This uncertainty can be removed by using a *four-terminal* resistor:

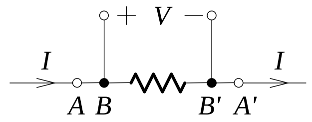



**Figure 10B.6**

One pair of terminals, $AA'$, is used to lead the current to and from the resistor. The voltage drop is measured between the other pair of terminals, $BB'$. The voltage $V$ is thus $I$ times the resistance from $B$ to $B'$, and does not include any contact drop that may be present at terminals $A$ and $A'$.

Resistors of low values are measured in terms of the resistance between the potential terminals, which thus becomes perfectly definite in value and independent of contact drop at the current terminals. (Contact drop at the potential terminals need not be a source of error, as the current crossing these contacts is usually extremely small – or even zero for null methods.)

## Measurement of Inductance

There are a few different bridges to measure inductance. The bridges not only measure the inductance of a real inductor, they also measure the resistance associated with a real inductor.

**Maxwell bridge**

The Maxwell bridge consists of the following arrangement:

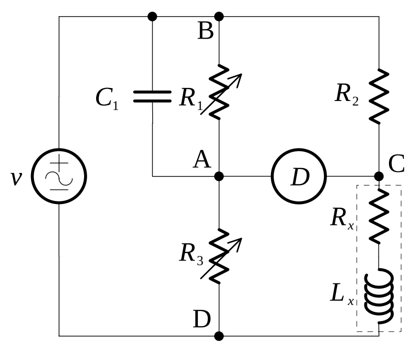



**Figure 10B.7**

At balance, $R_x$ and $L_x$ do not depend upon the frequency of the AC supply, thus eliminating a possible source of error. Another advantage is that it permits measurement of inductance in terms of capacitance. A capacitor can be made to have a more precise value than an inductor, since they effectively have no external field, are more compact, and are easier to shield.

A disadvantage is that it requires inconvenient large resistors to measure high $Q$ coils, and balancing $R_x$ and $L_x$ is iterative.





**Hay bridge**

The Hay bridge is similar to the Maxwell bridge, except the capacitor has a series resistance, instead of a parallel resistance:

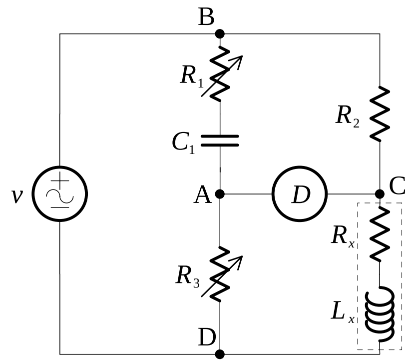



**Figure 10B.8**

This gives more convenient values of resistance and better balancing for high $Q$ coils. For high $Q$ coils, the frequency dependence is not a serious concern, because the terms involving frequency are small. For low $Q$ coils, the frequency is important, and it is better to use the Maxwell bridge.

## Measurement of Capacitance

The measurement of capacitance is carried out by a comparison bridge. This means we compare the value of the unknown capacitance with a known capacitance:

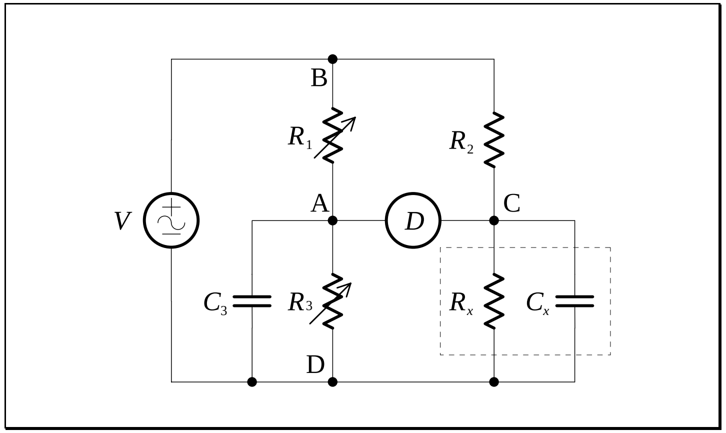



**Figure 10B.9**





## Average and RMS Values of Periodic Waveforms

Consider a current waveform $i(t)$ of period $T$. We define:

- Average or mean:

$$I_{AV} = \frac{1}{T}\int_0^T i(t)dt \tag{10B.12}$$

- RMS (root of mean of squares) or effective value (DC value producing same energy in time $T$):

$$W = \int_0^T pdt = \int_0^T Ri^2 dt = RI_{RMS}^2 T$$
$$I_{RMS} = \sqrt{\frac{1}{T}\int_0^T i^2 dt} \tag{10B.13}$$

- Form factor and crest factor:

$$ff = \frac{RMS}{AV} \tag{10B.14a}$$

$$cf = \frac{\text{peak}}{RMS} \tag{10B.6b}$$

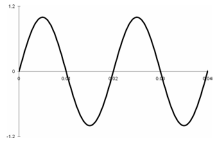
Any periodic wave can be decomposed into a sum of sine waves, with different amplitude and phase. This is a Fourier series:

$$i(t) = I_{AV} + \sum_{n=1}^{\infty} \hat{I}_n \cos(n\omega_0 t + \theta_n) \tag{10B.15}$$

$$I_{RMS} = \sqrt{I_{AV}^2 + \sum_{n=1}^{\infty} I_{nRMS}^2} \tag{10B.7b}$$

**Examples**

(i)



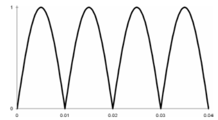
$$I_{AV} = 0$$

$$I_{RMS} = \sqrt{\frac{1}{T}\int_0^T \left(\hat{I}\sin\omega t\right)^2 dt}$$

$$= \hat{I}\sqrt{\frac{1}{2T}\int_0^T \left(1-\cos 2\omega t\right)dt}$$

$$= \frac{\hat{I}}{\sqrt{2}}$$

(ii)



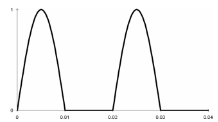
$$I_{AV} = \frac{\hat{I}}{T/2}\int_0^{T/2} \sin\omega t dt$$

$$= \frac{\hat{I}\omega}{\pi}\left[-\frac{\cos\omega t}{\omega}\right]_0^{\pi/\omega}$$

$$= \frac{2\hat{I}}{\pi}$$

$$I_{RMS} = I_{RMS} \text{ sine wave,} \quad ff = \frac{\pi}{2\sqrt{2}} = 1.11$$

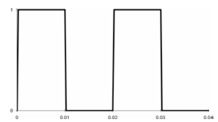
(iii)



$$I_{AV} = \frac{\hat{I}}{\pi}$$

$$I_{RMS} = \frac{\hat{I}}{2}$$

(iv)



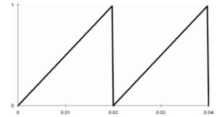
$$I_{AV} = \frac{\hat{I}}{2}$$

$$I_{RMS} = \frac{\hat{I}}{\sqrt{2}}$$





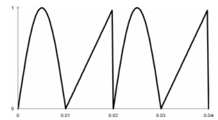
(v)



$$I_{AV} = \frac{\hat{I}}{2}$$

$$I_{RMS} = \sqrt{\frac{1}{T}\int_0^T \left(\frac{\hat{I}}{T}t\right)^2 dt} = \frac{\hat{I}}{\sqrt{3}}$$

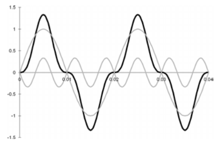
(vi)



$$I_{AV} = \hat{I}\left(\frac{1}{\pi}+\frac{1}{4}\right)$$

$$I_{RMS} = \hat{I}\sqrt{\frac{1}{4}+\frac{1}{6}}$$

(vii)

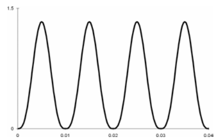


$$I_{AV} = 0$$

$$I_{RMS} = \sqrt{I_{1RMS}^2 + I_{3RMS}^2}$$

$$= \sqrt{\frac{\hat{I}_1^2}{2}+\frac{\hat{I}_3^2}{2}}$$

(vii)



$$I_{AV} = \frac{2\hat{I}_1}{\pi} + \frac{2\hat{I}_3}{3\pi}$$

$$I_{RMS} = \sqrt{I_{1RMS}^2 + I_{3RMS}^2}$$

$$= \sqrt{\frac{\hat{I}_1^2}{2}+\frac{\hat{I}_3^2}{2}}$$

## References


Jones, D. and Chin, A.: *Electronic Instruments and Measurements*, John Wiley & Sons, Inc., New York, 1983.

Stout, M.: *Basic Electrical Measurements 2nd Ed.*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1960.

# 10B.14

## Problems

**1.**

Derive the equation for the Kelvin bridge given by Eq. (10B.10).

**2.**

Derive equations for $R_x$ and $L_x$ in the Maxwell bridge.

**3.**

Derive equations for $R_x$ and $L_x$ in the Hay bridge.

# Lecture 11A – The Operational Amplifier

*The emitter-coupled differential amplifier. Common mode rejection ratio (CMRR). The operational amplifier.*

## The Emitter-Coupled Differential Amplifier

The emitter-coupled differential amplifier is the most important amplifier configuration in analog integrated circuits. It has two inputs and two outputs. Its usefulness is derived from one basic property of the amplifier: it amplifies the difference in voltage that may exist between its two input terminals. This property will turn out to be extremely desirable when we look at the operational amplifier, and the use of "feedback".

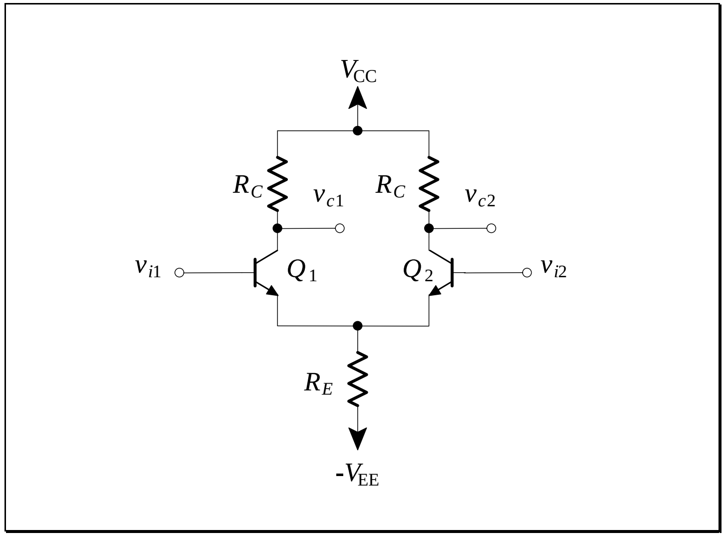



**Figure 11A.1**

For the circuit to work, it is critical that the two sides of the amplifier behave in exactly the same way. This means the transistors and resistors must be "matched" so that the circuit is perfectly symmetric.

**DC Conditions**

For DC conditions, the signals at the base are set (using superposition) to zero – the base of each transistor is tied to the common.

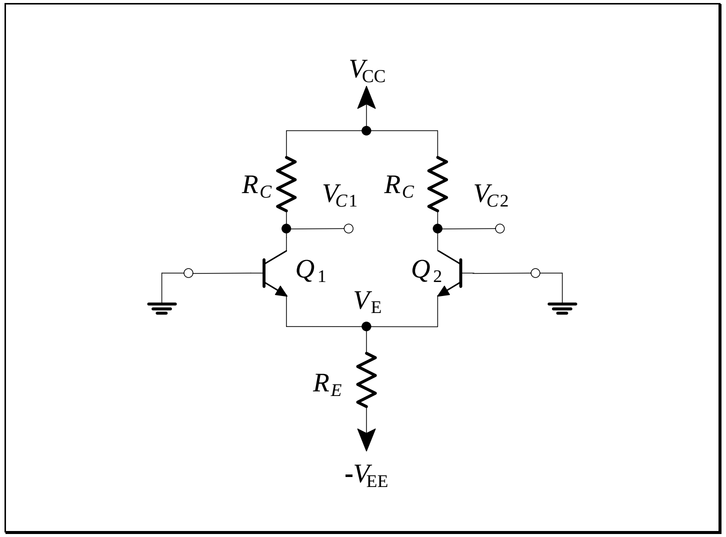



**Figure 11A.2**

Now assume that the transistors are conducting, so that there will be a voltage $V_E$ at the emitters.

$$V_E = V_B - V_{BE} \approx 0 - 0.7 = -0.7 \text{ V} \tag{11A.1}$$

Since the two transistors are matched, and since they have the same EBJ voltage, the emitter currents will be the same. Hence the collector currents will be the same, and the voltage at the collectors will be the same.

$$I_{CQ} \approx \frac{V_E + V_{EE}}{2R_E} \tag{11A.2a}$$

$$V_C = V_{CC} - R_C I_{CQ} \tag{11A.2b}$$

The output is taken differentially between the two collectors, so there will be no DC output voltage (0 V).





**AC Conditions**

Any two inputs we apply at the bases can be thought of as the superposition of two types of voltage: an average and a difference.

We call the average value of the two signals the common mode voltage, and the difference between the two signals the differential mode voltage:

$$\text{common mode voltage} \qquad v_{icm} = \frac{v_{i1} + v_{i2}}{2} \tag{11A.3a}$$

$$\text{differential mode voltage} \qquad v_{id} = v_{i1} - v_{i2} \tag{11A.3b}$$

The individual applied signals can be expressed as functions of the common mode voltage and differential mode voltage:

$$v_{i1} = v_{icm} + \frac{v_{id}}{2} \tag{11A.4a}$$

$$v_{i2} = v_{icm} - \frac{v_{id}}{2} \tag{11A.3b}$$

Any input signal we consider will be decomposed into these two types of signal, to simplify analysis of the circuit. We can find the gain of the amplifier for each type of signal, and use superposition to obtain the overall output voltage:

$$A_d = \text{differential mode OC voltage gain} \tag{11A.5a}$$

$$A_{cm} = \text{common mode OC voltage gain} \tag{11A.4b}$$

$$v_o = A_d v_{id} + A_{cm} v_{icm} \tag{11A.4c}$$

**Common Mode**

For small signals, we carry out ordinary circuit analysis using the small signal equivalent circuit for each BJT:

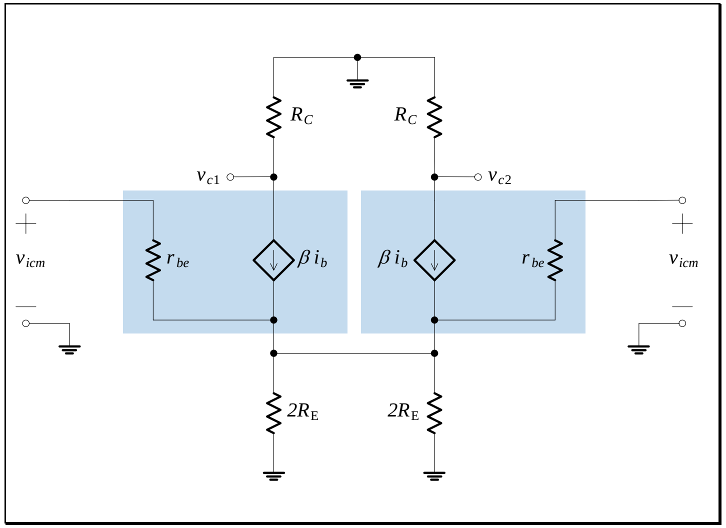



**Figure 11A.3**

Note that the resistor $R_E$ has been split into two parallel resistors of value $2R_E$. The transistors have exactly the same voltage applied across the EBJ, which implies that the collector currents must be identical. Because of the symmetry of the circuit, there is no current in the lead connecting the two emitters. The circuit behaviour is unchanged if this lead is removed. When this is done, the circuit reduces to two "half" circuits that are completely independent.





Only one of the half circuits need be analyzed to predict circuit behaviour:

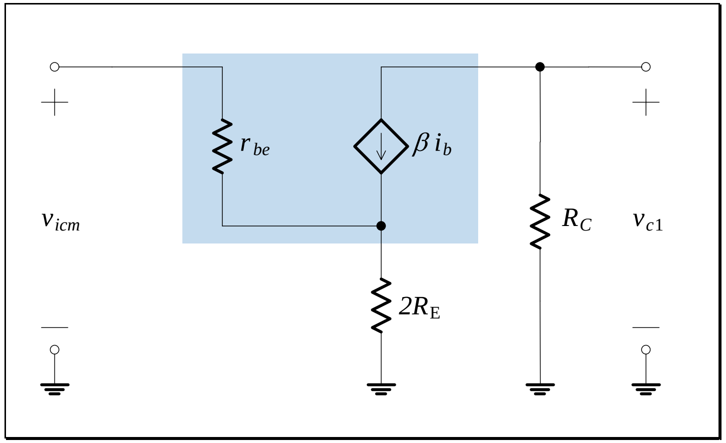



**Figure 11A.4**

KVL around the input gives:

$$v_{icm} - r_{be} i_b - 2R_E(\beta+1) i_b = 0$$

$$i_b = \frac{v_{icm}}{r_{be} + 2R_E(\beta+1)} \tag{11A.6}$$

The common mode output voltage is given by Ohm's law:

$$\begin{aligned} v_{c1} &= -R_C i_c = -R_C \beta i_b \\ &= v_{icm}\left[\frac{-\beta R_C}{r_{be} + 2R_E(\beta+1)}\right] \end{aligned} \tag{11A.7}$$

If the output is taken differentially, then the output common-mode voltage $v_o \equiv v_{c1} - v_{c2}$ will be zero. On the other hand, if the output is taken single-endedly (say, between the collector of $Q_1$ and common), then the common mode gain $A_{cm}$ will be finite and given by:

$$A_{cm} = \frac{v_{c1}}{v_{icm}} = -\frac{\beta R_C}{r_{be} + 2R_E(\beta+1)} \approx \frac{-R_C}{2R_E} \qquad \text{(11A.6)}$$

To make the common mode voltage gain small, we choose the resistor $R_E$ to be very large. Ideally, it would be infinite, and the transistors would be biased using a current source.

This common mode output voltage only appears when we take the output from a single collector to common (we eventually want our output signal referenced to common). If we take the output differentially, then the common mode voltage gain will be zero. Integrated circuits utilizing several differential pairs have the output of the first differential pair connected to the input of a second differential pair to increase rejection of common mode signals.

In real integrated circuits it is impossible to exactly match the components on both sides of the differential pair, so there will always be some finite (but small) common-mode gain, even if the output is taken differentially.





**Differential Mode**

We again carry out the usual circuit analysis using the small signal equivalent circuit model for the BJT:

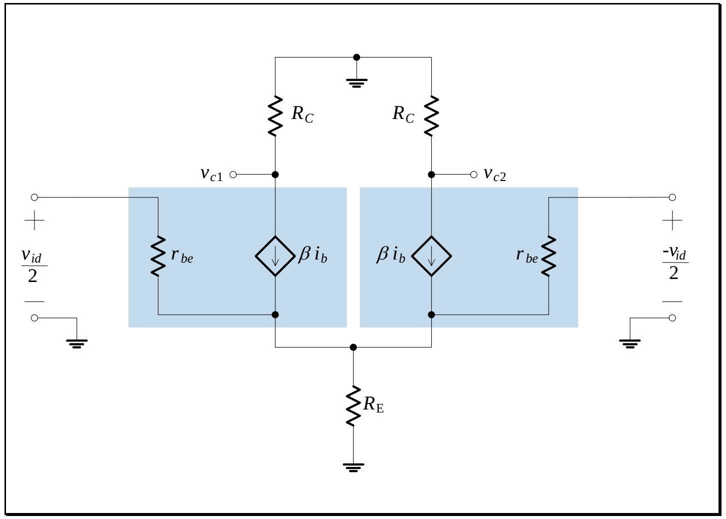



**Figure 11A.5**

By symmetry arguments the voltage at the emitters must be zero. To see this, use superposition. Let the voltage at the emitter be $v_{e1}$ due to the input $v_{id}/2$, with the other source set to zero. Then the voltage at the emitter due to the input $-v_{id}/2$ must be $-v_{e1}$ (the circuit is linear). The superposition of the two voltages leads to 0 V. Therefore, the resistor $R_E$ may be shorted without affecting circuit behaviour.

The equivalent circuit then becomes:

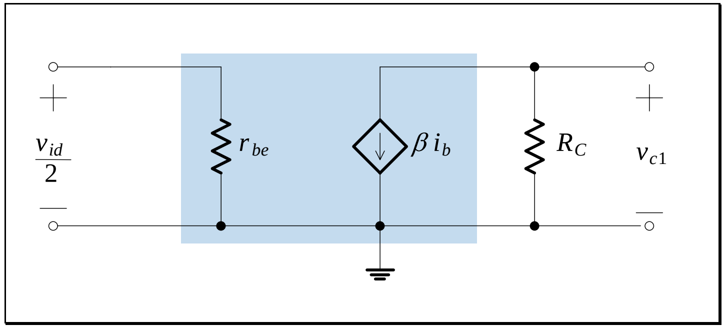



**Figure 11A.6**

KVL at the input gives:

$$\frac{v_{id}}{2} - r_{be} i_b = 0$$

$$i_b = \frac{v_{id}}{2r_{be}} \tag{11A.8}$$

The output voltage is given by Ohm's law:

$$v_{c1} = -\frac{\beta R_C}{2r_{be}} v_{id} \tag{11A.9}$$

If the output is taken differentially, then the differential gain of the differential amplifier will be:

$$A_d = \frac{v_{c1} - v_{c2}}{v_{id}} = -\frac{\beta R_C}{r_{be}} \tag{11A.10}$$





On the other hand, if we take the output single-endedly, then the differential gain will be given by:

$$A_d = \frac{v_{c1}}{v_{id}} = -\frac{\beta R_C}{2r_{be}} \tag{11A.11}$$

### Common-Mode Rejection Ratio

A measure of the differential pair's circuit quality is the common-mode rejection ratio. It is defined as:

$$CMRR = \frac{|A_d|}{|A_{cm}|} \tag{11A.12a}$$

$$CMRR\,(\text{dB}) = 20\log\frac{|A_d|}{|A_{cm}|} \tag{11A.12b}$$

Ideally, we would like $A_{cm}$ to be zero so that we have infinite common mode rejection.

For a perfectly matched differential pair, the common-mode rejection ratio is:

$$CMRR = \frac{|A_d|}{|A_{cm}|} \approx \frac{\beta R_C / 2r_{be}}{R_C / 2R_E} = \frac{\beta R_E}{r_{be}} = g_m R_E \tag{11A.13}$$

## The Operational Amplifier

Operational amplifiers are integrated circuit amplifiers consisting of several stages, fabricated within the top 10 μm of a 250 μm *Si* chip. Op amps amplify the voltage difference between the two inputs. External components are connected by conductive paths in the silicon. Internally DC coupling is used and bulky passive elements (capacitors) are avoided. The complete device is small, cheap, has very high gain, wide bandwidth and a large operating temperature range ($A \approx 2 \times 10^5$, $Z_i \approx 2\ \text{M}\Omega$, $Z_o \approx 75\ \Omega$).

BJT differential pairs are mostly used at the input because they are easy to match, have a good CMRR and are capable of carrying larger currents than FETs. However, some operational amplifiers have a FET input, with the rest of the circuit being made from BJTs.





The symbol of the operational amplifier is:

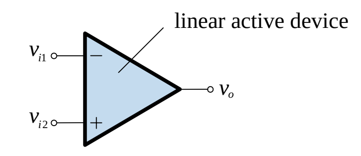



**Figure 11A.7**

The operational amplifier also has a common mode gain and a differential mode gain. We can use superposition to find the output for any given input:

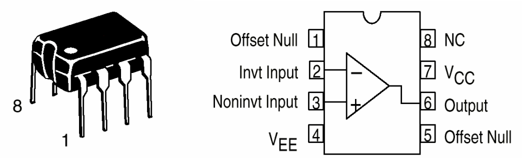
$$v_{i2} = 0 \quad v_o = \left( A_{dm} + \frac{A_{cm}}{2} \right) v_{i1} = A_1 v_{i1} \quad \text{(negative)} \tag{11A.14a}$$

$$v_{i1} = 0 \quad v_o = \left( -A_{dm} + \frac{A_{cm}}{2} \right) v_{i2} = A_2 v_{i2} \quad \text{(positive)} \tag{11A.14b}$$

Therefore, we generally call the inputs:

- 1 = inverting input (–)
- 2 = non inverting input (+)

Typical 8-pin package details are shown below:




TL071 package details

**Figure 11A.8**

## References

Sedra, A. and Smith, K.: *Microelectronic Circuits*, Saunders College Publishing, New York, 1991.

# Lecture 11B – Meters

*The moving iron meter. The electrodynamic meter (wattmeter). The ohmmeter. Electronic meters. The analog AC voltmeter. The differential voltmeter. The digital meter.*

## The Moving Iron Meter

There are two types of moving iron meter – attraction and repulsion type. The attraction type works by having one piece of soft iron attracted by the magnetic field produced by a current through a coil. The soft iron will be attracted to where the field is greatest.

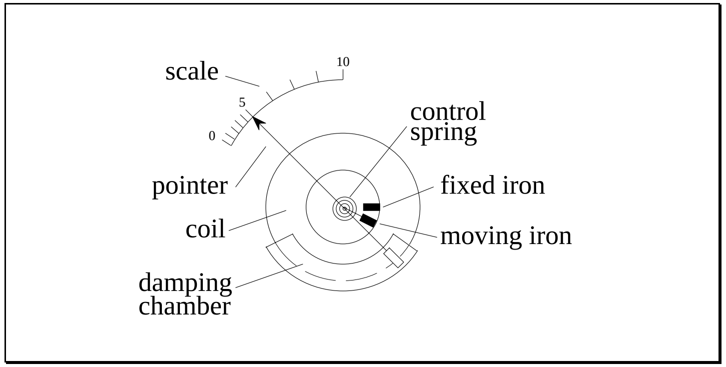



**Figure 11B.1**

For the repulsion type meter, two pieces of soft iron are placed inside a coil. Both fixed and moving irons are magnetized with the same polarity. The field is produced by the current $I$ being measured.

The deflecting torque is:

$$T_d = \frac{i^2}{2}\frac{dL}{d\theta} \tag{11B.1}$$

With alternating current, the torque fluctuates but is always in the same direction.

The movement takes a position determined by the average torque:

$$T_{d_{AV}} = \frac{1}{T}\int_0^T T_d dt = \frac{1}{2}\frac{dL}{d\theta}\frac{1}{T}\int_0^T i^2 dt$$
$$= \frac{1}{2}\frac{dL}{d\theta}\left(i^2\right)_{AV} \qquad (11B.2)$$

Therefore, the moving iron meter responds to $(i^2)_{AV}$ or $(i_{RMS})^2$ and reads RMS. It can be directly calibrated in RMS values. At balance:

$$T_d = T_r$$
$$\frac{i^2}{2}\frac{dL}{d\theta} = K_r \alpha \qquad (11B.3)$$

The shape of the irons is designed to give:

up to 10% of scale - $\frac{dL}{d\theta} = K_1 \quad \alpha = K' i^2$ (square law) (11B.4a)

rest of scale - $\frac{dL}{d\theta} = \frac{K_2}{\alpha} \quad \alpha = K'' i$ (linear scale) (11B.4b)

**Advantages**

- An RMS meter
- Simple
- Robust
- Cheap

**Disadvantages**

- Affected by frequency
- Affected by hysteresis (descending *I* or *V* readings > ascending readings)





**The Electrodynamic meter (wattmeter)**

The principle of operation of an electrodynamic meter is similar to a moving coil meter. All coils are air-cored (the main field $B_2$ is small, but there is no hysteresis). The fixed coils set up an almost uniform field, in which a moving coil is placed. The moving coil also has a current through it, and so the Lorentz force law applies.

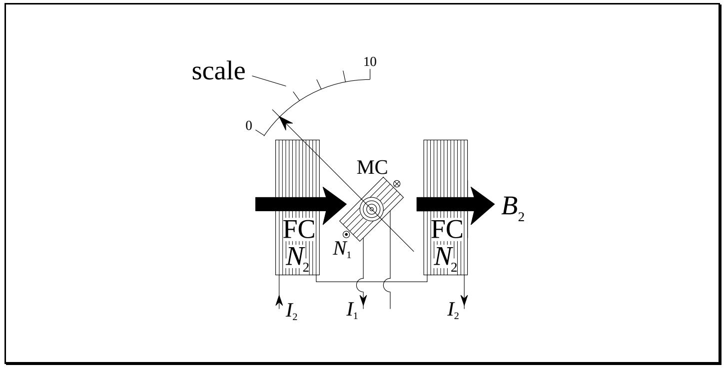



**Figure 11B.2**

If the moving coil radius is very small compared to the fixed coil radius, then the field at the moving coil, due to the fixed coils, is given by:

$$B_{12} = \frac{\mu_0 I_2 R_0^{\,2} N_2}{\left(R_0^{\,2} + D^2\right)^{3/2}} \qquad (11B.5)$$

where the various dimensions are:

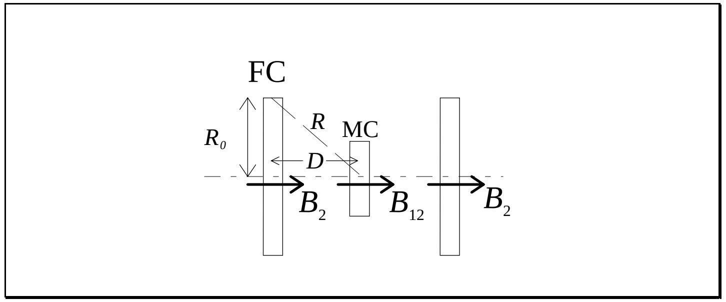



**Figure 11B.3**

The deflecting torque is then:

$$T_d = KI_1 B_{12} \cos\theta = K_d I_1 I_2 \cos\theta \tag{11B.6}$$

**Ammeter**

To use the meter as an ammeter, connect the moving coil and fixed coils in series. Then $I_1$ and $I_2$ are equal and:

$$T_{d\,AV} \propto \left(i^2\right)_{AV} \tag{11B.7}$$

**Voltmeter**

A voltmeter can be made from any ammeter by placing a large resistor in series with it to limit the current. Therefore it will be an RMS voltmeter.





**Electromechanical Wattmeter**

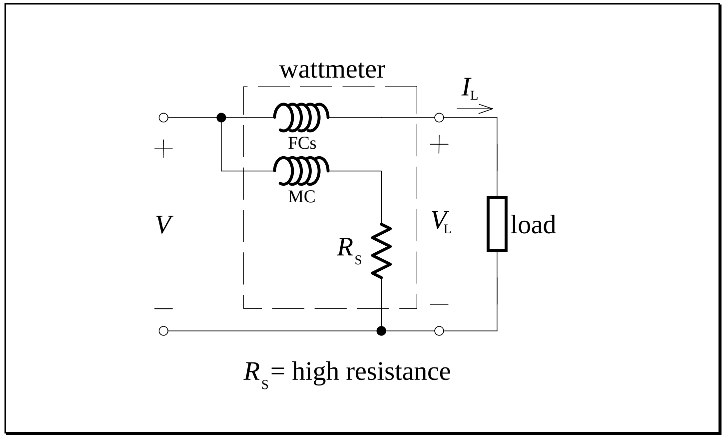



**Figure 11B.4**

If the coils are connected as shown then:

$$I_2 = I_L \tag{11B.8a}$$

$$I_1 = \frac{V}{R_{MC} + R_S} \propto V \tag{11B.8b}$$

Also:

$$\alpha_{AV} \propto \left(I_1 I_2\right)_{AV} \tag{11B.9a}$$

$$\alpha_{AV} \propto \left(VI_L\right)_{AV} \tag{11B.9b}$$

But:

$$\begin{aligned} V &= V_L + R_F I_L \\ VI_L &= V_L I_L + R_F I_L^{\,2} \\ &= \text{load power} + \text{fixed coil heat loss} \end{aligned} \tag{11B.10}$$

Therefore, as connected, $\alpha_{AV} \propto (\text{load power} + \text{fixed coil heat loss})$. With the moving coil connected across just the load, the meter reads (load power + moving coil heat loss).

For sinusoidal voltages and currents:

$$v = \hat{V}\cos\omega t$$
$$i = \hat{I}_L\cos(\omega t + \theta)$$
$$\alpha \propto \frac{\hat{V}\hat{I}_L}{2}\{\cos\theta + \cos(2\omega t + \theta)\}$$
$$\alpha_{AV} \propto \frac{\hat{V}\hat{I}_L}{2}\cos\theta = V_{RMS}I_{L_{RMS}}\cos\theta \qquad (11B.11)$$

### The Ohmmeter

An ohmmeter consists of a moving coil meter, a battery, and a number of resistors that determine the range of the measurement.

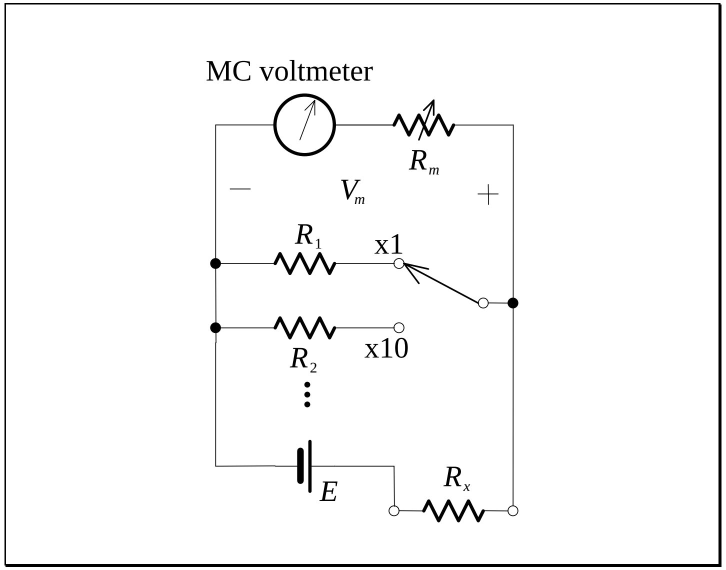



**Figure 11B.5**





$R_m$ is the moving coil resistance, plus the current limiting resistance, plus the zero adjust resistance. It limits the current in the moving coil to full scale deflection (FSD) current $I_{FS}$ when the test leads are shorted. Therefore:

$$R_m = \frac{E}{I_{FS}} \qquad (11B.12)$$

Also:

$$V_m = \frac{R_1 \parallel R_m}{R_1 \parallel R_m + R_{x1}}E \qquad (11B.13)$$

The lowest range (x1, say) shunt resistance $R_1$ is chosen so that a specific resistance $R_{x1}$ gives half scale deflection (HSD):

$$V_m = \frac{I_{FS}}{2}R_m = \frac{E}{2} \qquad (11B.14)$$

or, from Eq. (11B.13):

$$R_{x1} = R_1 \parallel R_m$$
$$R_1 = \frac{R_{x1}R_m}{(R_m - R_{x1})} \qquad (11B.15)$$

To determine the shunt $R_3$ for the x100 range we use the previous equation with $R_{x3} = 100R_{x1}$.

$R_m$, $R_1$, $R_2$, ... and $E$ are all constant. The meter deflection, using Eq. (11B.13), is:

$$\alpha = KV_m = \frac{K_1}{K_2 + R_x} \propto \frac{1}{R_x} \qquad (11B.16)$$

Therefore, the meter scale is not uniform.

## Electronic Meters

Some of the existing type of electronic meter are:

Analog – quantity to be measured is converted to DC or DC proportional to heating energy of input wave, which then drives a DC moving coil electromechanical meter (usually taut-band suspension).

Differential – quantity to be measured is compared to a reference voltage.

Digital – analog input is converted to digital form. The display is also digital.

In all cases the basic "building blocks" are:

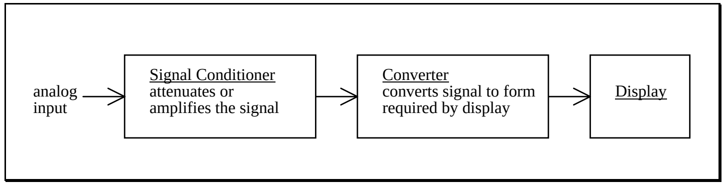



**Figure 11B.6**

## The Analog AC Voltmeter

A basic AC voltmeter circuit is the following:

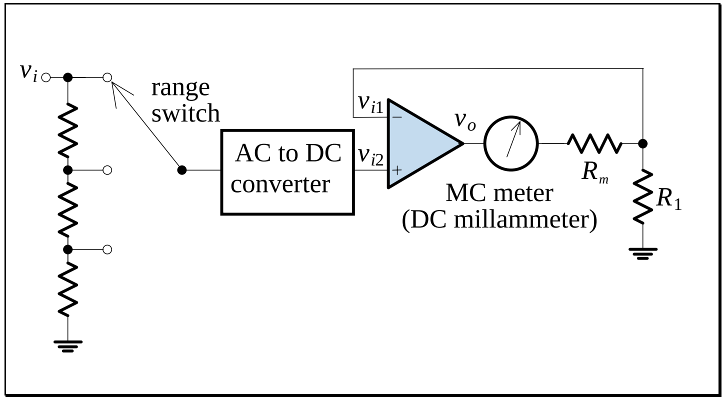



**Figure 11B.7**





The input consists of a series connection of resistors. A voltage is selected from this potential divider using a physical switch that corresponds to a voltage range. The input resistance is seen to be a constant, no matter where the switch is positioned.

The voltage waveform is then applied to an AC to DC converter circuit of some sort. The DC voltage resulting from the conversion is applied to the noninverting terminal of an op amp so as not to load the measured circuit.

A moving coil meter (which responds to the average) is placed in the feedback loop of the amplifier. In order to stabilize the output, a proportion of the output is fed back to the input. The operational amplifier has a very high gain, very high input impedance and a very low output impedance.

The meter scale can be calibrated in terms of the RMS value of a sine wave. The meter response is given by:

$$V_{i1} = \frac{R_1}{R_1 + R_m} V_o = \beta V_o$$

$$V_o = A\left(V_{i2} - \beta V_o\right)$$

$$= \frac{A}{1 + A\beta} V_{i2} \qquad (11B.17a)$$

$$V_m = \frac{R_m}{R_m + R_1} V_o \approx \frac{R_m}{R_1} V_{i2} \qquad (11B.15b)$$

Some basic AC to DC converter circuits are:

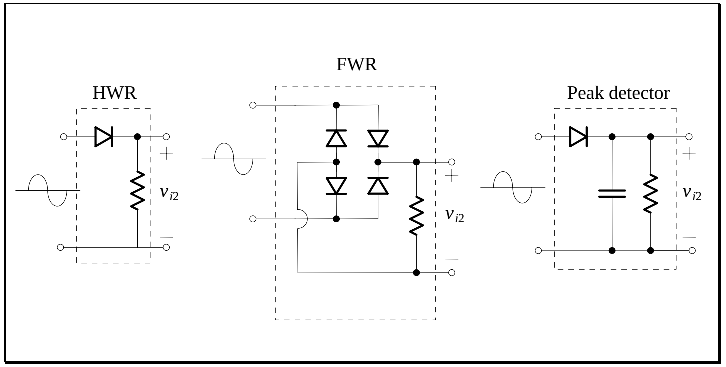



**Figure 11B.8**

For the full wave rectifier converter circuit, the moving coil meter reads $1.11V_{\mathrm{AV}}$. For the peak detector, it reads $0.707V_{\mathrm{AV}}$.

If the signal needs to be amplified, then rectification takes place after amplification:

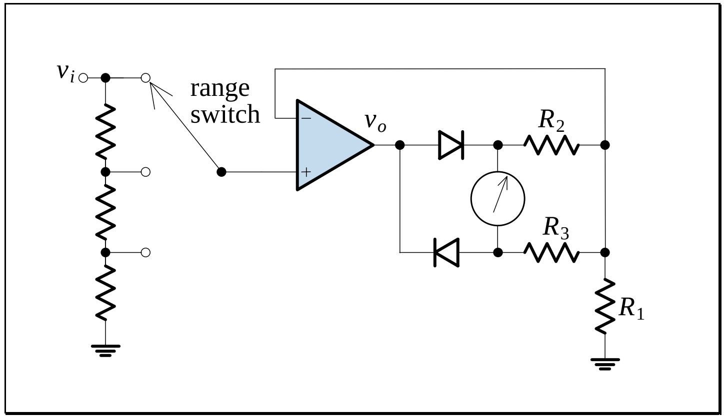



**Figure 11B.9**

The diodes are placed in the feedback loop so that their nonlinear characteristic is of no consequence.





## The Differential Voltmeter

The differential voltmeter compares the unknown voltage to a standard reference voltage by using a precision divider:

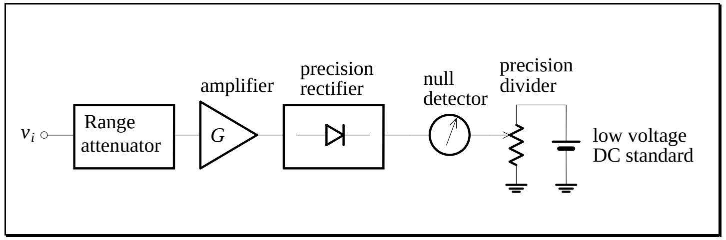



**Figure 11B.10**

It is accurate but slow.

## The Digital Meter

Digital instruments have a good readability (not prone to human error) and are more accurate than analog meters. They have a greater resolution and thus have less ranges than analog meters (they are often auto ranging).

A digital meter contains digital circuitry to obtain a measurement. The converter block of the general electronic meter is therefore:

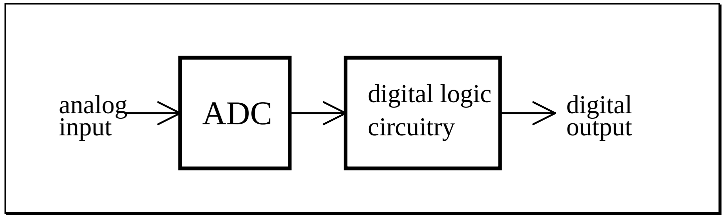



**Figure 11B.11**

There are two main types of analog to digital converter (ADC), integrating and non-integrating. Integrating ADCs are mainly associated with discrete circuitry. Non-integrating ADCs are more common in integrated circuits (ICs) and microcontrollers.

**Integrating ADC**

One example of an integrating ADC is the voltage to frequency integrating ADC (relatively slow, but accurate):

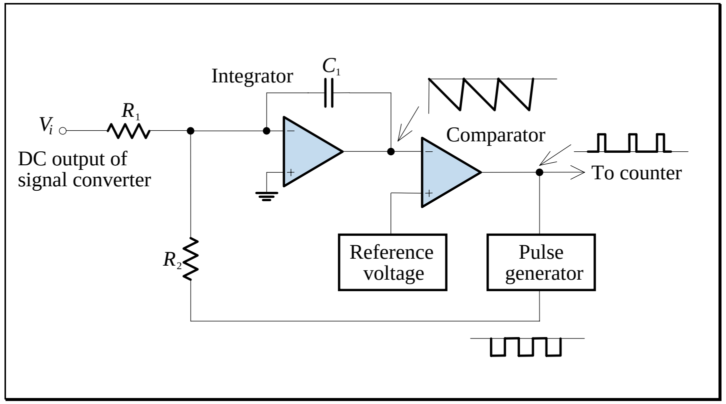



**Figure 11B.12**

With a DC input to an integrator, we have:

$$v_o = -\frac{1}{R_1 C_1}\int_0^t v_i dt$$
$$= -\frac{V_i}{R_1 C_1} t + v_o(0) \qquad (11B.18)$$

Thus the output, for a DC input, is a negative ramp.

When $v_o$ reaches a certain negative level, the comparator triggers the pulse generator which generates a negative voltage step with magnitude > $|V_i|$ and the integrator output is zeroed. The comparator detects $v_o < v_{ref}$ and turns off the pulse generator.

The process is repeated. The rate of pulse generation is governed by the magnitude of the DC input ($V_i$). A larger input causes a steeper ramp and a higher pulse rate. The comparator output waveform is fed to a digital counter and the pulse rate (calibrated in volts) is displayed.





Noise tends to average out (the integral of noise is generally zero).

**Microprocessor Controlled**

A microprocessor based instrument is an "intelligent" instrument. The microprocessor can control such things as a keypad, display, the ADC; other internal circuitry such as oscillators and frequency dividers; the nature of the measurement, e.g. from voltmeter to ohmmeter, etc.

Most of these functions can be incorporated into one integrated circuit (which is mass produced), known as a microcontroller. These meters are therefore cheaper than analog meters.

A disadvantage of microprocessor-based meters is that they can only measure signals below a certain frequency, known as the “foldover frequency”, which is intimately related to how fast digital samples of the analog waveform can be taken. However, with advances in computing speed, this is becoming less of a concern.

## References

Jones, D. and Chin, A.: *Electronic Instruments and Measurements*, John Wiley & Sons, Inc., New York, 1983.

# Lecture 12A – Op-Amp Circuits

*An op-amp model. The ideal op-amp. The inverting amplifier. The noninverting amplifier. The voltage follower. An adder circuit. Integrator circuits. A precision half wave rectifier.*

## An Op-Amp Model

A model of an op-amp is shown below:

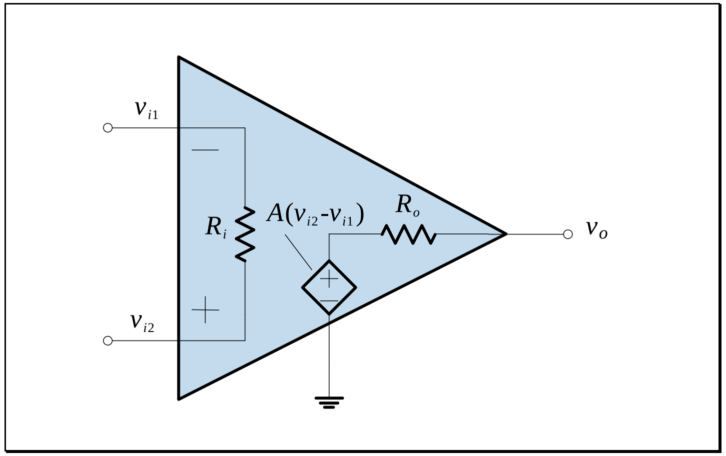



**Figure 12A.1**

Typical values for the op-amp model elements are as follows:

$$R_i = 1\,\mathrm{M}\Omega, \quad R_o = 75\,\Omega, \quad A = 2\times 10^5 \tag{12A.1}$$

To analyze the circuit, we will make three assumptions:

- Since $R_i$ is very large, we will assume it to be infinite.
- Since $R_o$ is very small, we will assume it be zero.
- Since $A$ is very large, we will assume it to be infinite.

These assumptions lead to the model for an ideal o-amp.

## The Ideal Op-Amp

An ideal op-amp has the following parameter values:

The ideal op-amp's parameters

$$A = \infty$$
$$R_i = \infty$$
$$R_o = 0 \tag{12A.2}$$

If there is a *negative* feedback path (i.e. a connection between the output of the amplifier and the inverting input terminal), then the op-amp will have a finite output voltage. It follows that:

$$v_o = A(v_{i2} - v_{i1})$$
$$v_{i2} - v_{i1} = \frac{v_o}{A} = \frac{v_o}{\infty} = 0 \tag{12A.3}$$

Therefore:

The *virtual* short circuit is the key to analysing op-amp circuits

$$v_{i1} = v_{i2} \tag{12A.4}$$

The input to the op-amp looks like a short circuit for voltages, but due to the input resistance being infinite, it looks like an open circuit for currents. The input terminals can therefore be considered a *virtual short circuit*. We will use the virtual short circuit concept frequently.





## The Inverting Amplifier

The inverting amplifier circuit is shown below:

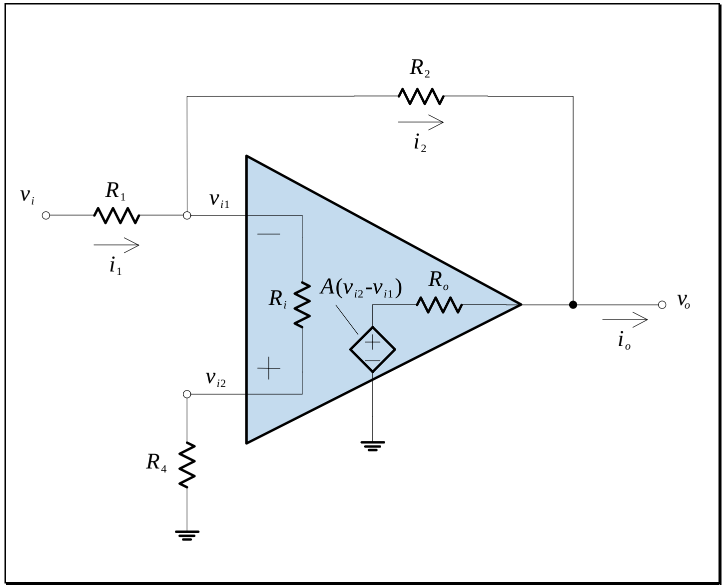



**Figure 12A.2**

It is called inverting because the output will be inverted (which implies a negative gain).

The virtual short circuit concept is very important because it allows us to analyze a circuit very quickly. To see this, we will analyze the circuit using the ideal op-amp model and the concept of a virtual short circuit, and then see how things change with a finite value of *A*.

With a virtual short circuit, the inverting amplifier becomes:

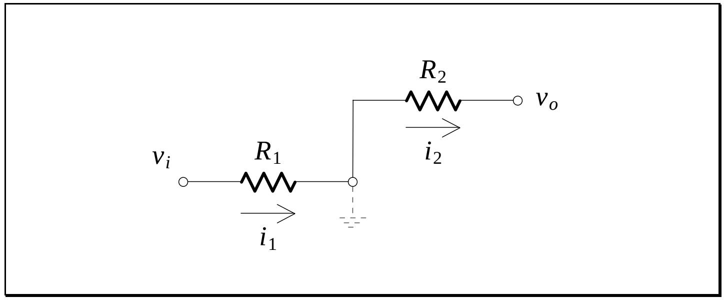



**Figure 12A.3**

### Gain

There is only one current, so:

$$i_1 = \frac{v_i}{R_1} = -\frac{v_o}{R_2} \tag{12A.5}$$

The voltage gain is then:

$$A_v = \frac{v_o}{v_i} = -\frac{R_2}{R_1} \tag{12A.6}$$

### Input Resistance

With a virtual short circuit, the input resistance is obtained by inspection:

$$R_{\text{in}} = \frac{v_i}{i_1} = R_1 \tag{12A.7}$$

### Output Resistance

With a virtual short circuit as in Figure 12A.3, the output voltage is independent of any load resistor. Thus, the output resistance of the amplifier is:

$$R_{\text{out}} = 0 \tag{12A.8}$$





### Effect of Finite Open-Loop Gain

Firstly, we seek a representation of the circuit which corresponds to the Thévenin equivalent:

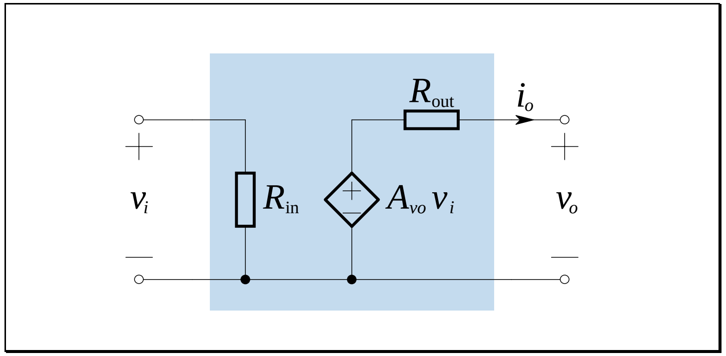



**Figure 12A.4**

With an assumed output current as shown in the figure (the load is not shown), we can write KVL and get:

$$v_o = A_{vo} v_i - R_{\text{out}} i_o \tag{12A.9}$$

We now seek an expression for the output of the real circuit that is in a similar form so that we can identify the open-circuit voltage gain and output resistance by inspection.

To include the effect of a finite $A$, we first introduce the *feedback factor*:

$$\beta = \frac{R_1}{R_1 + R_2} \tag{12A.10}$$

Note that:

$$1 - \beta = \frac{R_2}{R_1 + R_2} \tag{12A.11}$$

which is an expression that will be used later.

With reference to Figure 12A.2, we now perform KCL at the input:

$$\frac{v_{i1}-v_i}{R_1}+\frac{v_{i1}-v_o}{R_2}=0 \tag{12A.12}$$

Multiplying through by $R_1R_2$ and rearranging, we get:

$$\begin{aligned} v_{i1} &= \frac{R_2}{R_1+R_2}v_i+\frac{R_1}{R_1+R_2}v_o \\ &= (1-\beta)v_i+\beta v_o \end{aligned} \tag{12A.13}$$

KCL at the output gives:

$$\frac{v_o-v_i}{R_1+R_2}+\frac{v_o-(-Av_{i1})}{R_o}+i_o=0 \tag{12A.14}$$

Substituting the expression for $v_{i1}$ from Eq. (12A.13) we get:

$$\frac{v_o-v_i}{R_1+R_2}+\frac{v_o+A(1-\beta)v_i+A\beta v_o}{R_o}+i_o=0 \tag{12A.15}$$

Rearranging we get:

$$v_o\left(\frac{1}{R_1+R_2}+\frac{1+A\beta}{R_o}\right)=\left(\frac{1}{R_1+R_2}-\frac{A(1-\beta)}{R_o}\right)v_i-i_o \tag{12A.16}$$

If we now let:

$$\frac{1}{R_{\text{out}}}=\frac{1}{R_1+R_2}+\frac{(1+A\beta)}{R_0} \tag{12A.17}$$





then we can see that the output resistance is composed of two resistances in parallel:

$$R_{\text{out}}=(R_1+R_2)\,||\,[R_0/(1+A\beta)] \tag{12A.18}$$

This is usually a very small resistance. As an example, an inverting amplifier with a nominal gain of $-R_2/R_1=-10$ ($\beta=1/11$) designed using an op-amp with $R_o=100\,\Omega$ and $A=10^5$ will have $R_{\text{out}}\approx 11\,\text{m}\Omega$.

We can now rewrite Eq. (12A.16) using our new definition of $R_{\text{out}}$:

$$v_o=\left(\frac{1}{R_1+R_2}-\frac{A(1-\beta)}{R_o}\right)R_{\text{out}}v_i-R_{\text{out}}i_o \tag{12A.19}$$

This is similar to Eq. (12A.9). By comparison, we can easily identify the open-circuit voltage gain:

$$\begin{aligned} A_{vo} &= \left(\frac{1}{R_1+R_2}-\frac{A(1-\beta)}{R_o}\right)R_{\text{out}} \\ &= \left(\frac{R_o-A(1-\beta)(R_1+R_2)}{(R_1+R_2)R_o}\right)\frac{(R_1+R_2)R_o/(1+A\beta)}{R_1+R_2+R_o/(1+A\beta)} \\ &= \frac{R_o-A(1-\beta)(R_1+R_2)}{(R_1+R_2)(1+A\beta)+R_o} \\ &= \frac{R_o-AR_2}{R_1+R_2+AR_1+R_o} \\ &= \frac{-R_2+R_o/A}{R_1+(R_1+R_2+R_o)/A} \end{aligned} \tag{12A.20}$$

With large open-loop gain *A*, the gain expression reduces to $A_v=-R_2/R_1$.

To obtain the input resistance, we do KCL at the inverting terminal:

$$i_1 = \frac{v_{i1} - v_o}{R_2} = \frac{v_{i1} - A(0 - v_{i1})}{R_2} = \frac{v_{i1}(1 + A)}{R_2}$$

$$\frac{v_{i1}}{i_1} = \frac{R_2}{1 + A} \qquad (12A.21)$$

and so:

$$R_{\text{in}} = R_1 + \frac{R_2}{1 + A} \qquad (12A.22)$$

With large open-loop gain *A*, the input resistance reduces to $R_{\text{in}} = R_1$.





**Resistor** $R_4$

Why do we need to put resistor $R_4$ on the noninverting terminal of the op-amp, if it is at common potential and carries no current?

The standard input stage of integrated circuit op-amps is a DC coupled differential amplifier. As there are no input coupling capacitors, a proportion of the DC bias currents circulate in the input and feedback circuits. To find the DC output voltage of the closed-loop amplifier due to the input bias current, we set the signal source to zero and obtain the circuit below:

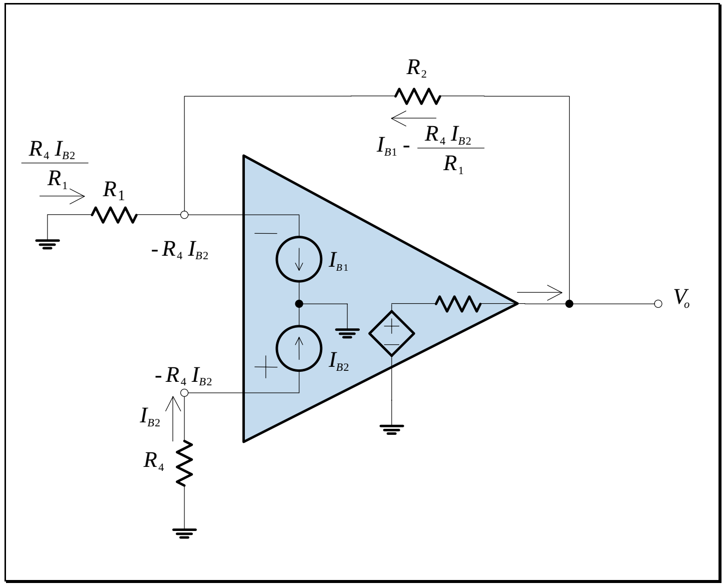



**Figure 12A.5**

With $R_4 = 0$, the output DC voltage is given by:

$$V_o = R_2 I_{B1} \qquad (12A.23)$$

which can be significant if $R_2$ is large.

With $R_4 \neq 0$, the output DC voltage is given by:

$$V_o = -R_4 I_{B2} + R_2\left(I_{B1} - R_4/R_1\, I_{B2}\right) \tag{12A.24}$$

For the case $I_{B1} = I_{B2} = I_B$, we get:

$$V_o = \left[R_2 - R_4\left(1 + R_2/R_1\right)\right] I_B \tag{12A.25}$$

We may reduce $V_o$ by selecting $R_4$ such that:

$$R_4 = \frac{R_2}{1 + R_2/R_1} = \frac{R_2 R_1}{R_1 + R_2} = R_1 \parallel R_2 \tag{12A.26}$$

Therefore, to reduce the effect of DC bias currents, we should select $R_4$ to be equal to the parallel equivalent of $R_1$ and $R_2$. Having selected this value, substitution into Eq. (12A.24) gives:

$$V_o = R_2\left(I_{B1} - I_{B2}\right) \tag{12A.27}$$

If we define the *input offset current* as the difference between the two input bias currents:

$$I_{\text{off}} = I_{B1} - I_{B2} \tag{12A.28}$$

then:

$$V_o = R_2 I_{\text{off}} \tag{12A.29}$$

which is usually about an order of magnitude smaller than the value obtained without $R_4$.





### The Noninverting Amplifier

The noninverting circuit has the input signal connected to the noninverting terminal of the op-amp in some way:

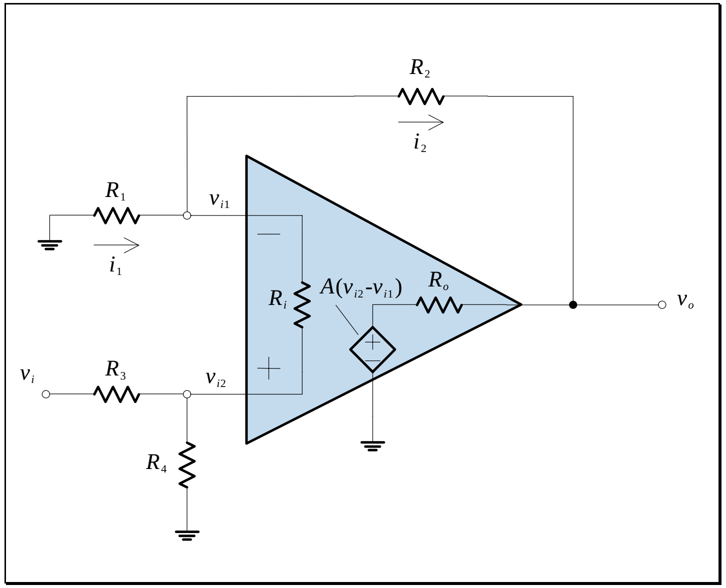



**Figure 12A.6**

An analysis of this circuit follows. First the gain:

$$v_{i2} \approx v_{i1}$$

$$\frac{R_4}{R_3 + R_4} v_i \approx \frac{R_1}{R_1 + R_2} v_o$$

$$\frac{v_o}{v_i} \approx \frac{1 + R_2/R_1}{1 + R_3/R_4} \tag{12A.30}$$

The input and output resistance are obtained in the same way as before:

$$R_{\text{in}} \approx R_3 + R_4 \tag{12A.31a}$$

$$R_{\text{out}} \approx 0 \tag{12A.26b}$$

## The Voltage Follower

The voltage follower is used instead of the source or emitter follower where precision is required. It is used for impedance matching.

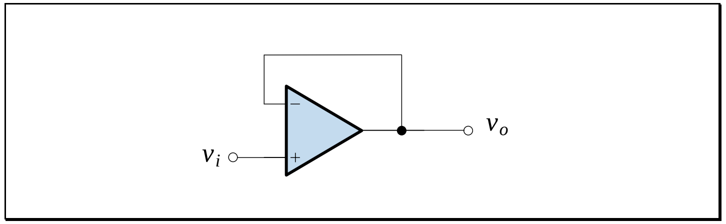



**Figure 12A.7**

To analyze the circuit we note that:

$$R_2 = R_3 = 0 \quad R_1 = R_4 = \infty \tag{12A.32a}$$

$$\therefore A_v \approx 1 \quad R_{\text{in}} \approx \infty \quad R_{\text{out}} \approx 0 \tag{12A.27b}$$

The voltage follower is used to provide isolation between two parts of a circuit when it is required to join them without interaction. For example, to couple a high resistance source to a low resistance load, without suffering a voltage drop, we insert a buffer between source and load:

A buffer is used to couple a high impedance to a low impedance

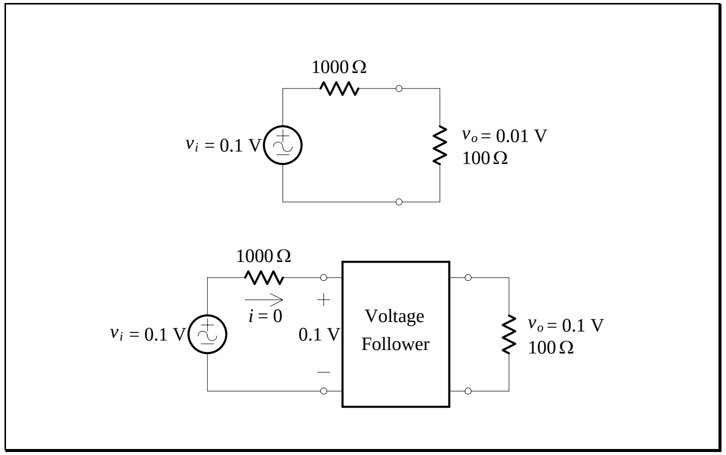



**Figure 12A.8**





## An Adder Circuit

An adder circuit performs the mathematical operation of addition on two (or more) voltages (hence the name operational amplifier):

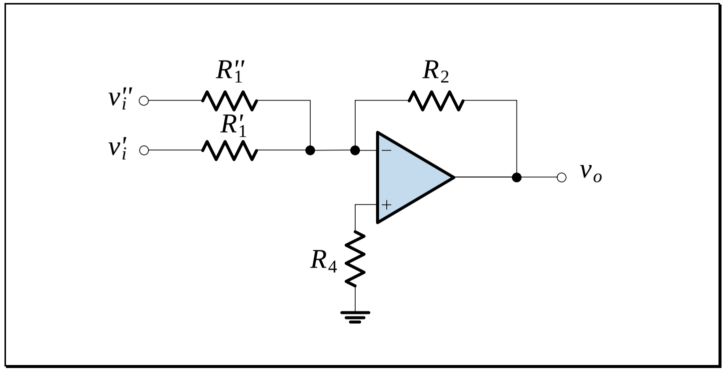



**Figure 12A.9**

We can use superposition, and the concept of the virtual short circuit, to obtain the gain of this circuit:

$$v_o = -R_2\left(\frac{v_i'}{R_1'} + \frac{v_i''}{R_1''}\right) \tag{12A.33}$$

## Integrator Circuits

The ideal integrator circuit is:

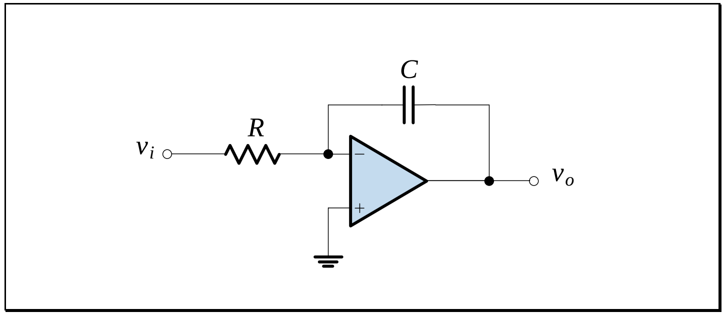



**Figure 12A.10**

We use the virtual short circuit concept to analyze the gain (it is essentially the same analysis as for the inverting amplifier). KCL at the inverting terminal gives:

$$\frac{v_i}{R} = -C\frac{dv_o}{dt}$$

$$v_o = -\frac{1}{RC}\int_0^t v_i dt + v_o(0) \qquad (12A.34)$$

Unfortunately, this circuit suffers from the fact that any DC at the input, such as the inherent input offset voltages and currents of the op-amp, will be integrated and eventually cause the output of the op-amp to saturate.





A practical circuit that alleviates this problem is known as the Miller integrator, shown below:

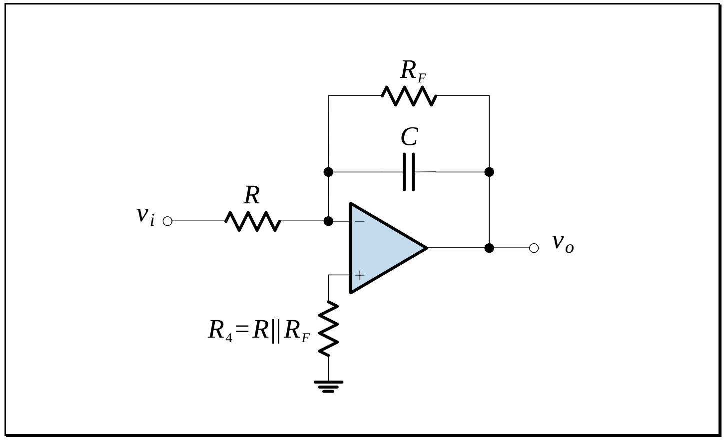



**Figure 12A.11**

The Miller integrator provides a path for the DC offset currents $V_{OS}/R$ and $I_{OS}$, with the result that the output has a DC component given by:

$$V_o = (1 + R_F/R)V_{OS} + R_F I_{OS} \qquad (12A.35)$$

To keep the DC offset at the output of the integrator low, we should select a small $R_F$. Unfortunately, however, the lower the value of $R_F$, the less ideal the integrator becomes. Thus selecting a value for $R_F$ is a trade-off between DC performance and integrator performance.

## A Precision Half Wave Rectifier

A precision half wave rectifier gets around the problem of the forward bias voltage drop with real diodes. We can rectify signals less than 0.7 V with this circuit:

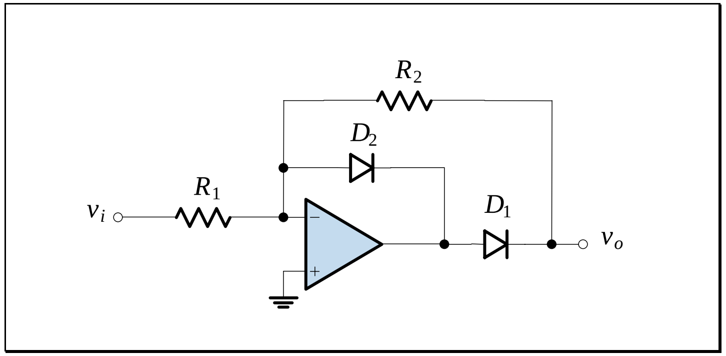



**Figure 12A.12**

To analyze this circuit, we firstly remember that the diode is a nonlinear element so that linear circuit analysis does not apply. We assume that somehow the op-amp is working so that there is a virtual short circuit at its input.

In the positive half cycle, with the op-amp working, the current is to the right in resistor $R_1$. The current cannot enter the op-amp inverting terminal, due to the infinite input resistance of the op-amp, so it must go up. Diode D2 is in the right direction to conduct this current, so it will. Since the diode is conducting, the voltage at the output terminal of the op-amp will be about -0.7 V. With D1 assumed off, the voltage across it is -0.7 V so our assumption is correct. The resistor $R_2$ does not conduct any current, so the voltage drop across it is also zero. The output of the circuit is therefore 0 V for a positive half cycle.





In the negative half cycle, with the op-amp working, the current is to the left in resistor $R_1$. The current cannot be coming from the op-amp inverting terminal, because it is like an open circuit, so it must be coming from the feedback circuit. Diode D1 is in the right direction to conduct this current, so it will, Since the diode is conducting, the voltage at the output of the circuit will be:

$$v_o = -\frac{R_2}{R_1} v_i \tag{12A.36}$$

This is determined solely by the external resistors. The voltage at the output terminal of the op-amp will be whatever it has to be to supply this current through D1. For example, it may be 0.7 V above the output voltage for large input signals, or it may be 0.5 V above the output voltage for small input signals.

The voltage across D2 is such as to reverse bias it. The purpose of D2 is to let the op-amp create a virtual short circuit for the positive half cycle. Otherwise the op-amp would saturate, and the output would not be zero.

## References

Sedra, A. and Smith, K.: *Microelectronic Circuits*, Saunders College Publishing, New York, 1991.

# Lecture 12B – Revision

*Identify essential material and problems.*

## Lecture 7A - essential

- MOSFET operation (qualitative)
- Basic amplifier circuit
- DC analysis using output characteristic (*Q*-point)

You should be able to determine the *Q*-point of a MOSFET from either the output characteristic or transfer characteristic, given nay biasing circuit.

## Lecture 7B - essential

- Transformer model (including magnetizing branch)
- Measurement of transformer parameters (SC and OC test)
- Current and Voltage Excitation

You should be able to calculate transformer model parameters using the short circuit and open circuit tests. You should be able to apply the concepts of current and voltage excitation to any magnetic system.

## Lecture 8A - essential

- Small signal equivalent circuit of the MOSFET (at low frequencies)
- Design of the common source amplifier
- Analysis of the source follower

You should be able to design a common source amplifier, using a variety of techniques, e.g. the output characteristic and load line, or the transfer characteristic. You should be able to apply the small signal equivalent circuit of a MOSFET to any circuit.

## Lecture 8B - essential

- The force equation for magnetic and electric systems.

You should be able to apply the force equation to any linear magnetic or electric system with moveable parts.

## Lecture 9A - essential

- BJT operation (qualitative)
- Design of a biasing circuit using a single power supply
- Small signal equivalent circuit of a BJT
- Common emitter amplifier
- The emitter follower

You should be able to design the bias circuit for an *npn* BJT using a single power supply. You should be able to analyze (determine input impedance, gain, output impedance) any BJT amplifier circuit, by using the small signal equivalent circuit. You should be able to obtain the frequency response of the gain for any amplifier circuit.

## Lecture 9B - essential

- Generator principle
- Motor principle
- The moving coil meter

You should be able to determine the deflection angle of a moving coil meter for any type of current.





## Lecture 10A - essential

- Voltage amplifier equivalent circuit
- Current amplifier equivalent circuit
- The decibel
- Frequency response

You should be able to obtain the equivalent circuit of any amplifier in terms of the above models. You should be able to preform an analysis of the frequency response of any amplifier.

## Lecture 10B - essential

- General bridge equations
- Average and RMS values of periodic waveforms

You should be able to derive the balance conditions for any bridge. You should be able to calculate the average and RMS values of any simple periodic waveform.

## Lecture 11A - general knowledge

Know that an operational amplifier simply amplifies the voltage difference between its two input terminals.

## Lecture 11B - essential

- The moving iron meter

You should be able to determine the deflection angle of a moving iron meter for any type of current. Be aware of other meter movements and construction.

## Lecture 12A - essential

- The inverting amplifier
- The noninverting amplifier
- The voltage follower
- The adder circuit
- The integrator
- The precision HWR

You should be able to derive the gain for any linear op-amp circuit. You should be able to analyze an op-amp circuit that contains diodes, and describe the circuit's function.

## General

All problems should have been completed, except those that have an asterisk.

All examples in the notes should be read and understood.

Permanent magnets, having been left out of the mid, may be in the final.

## Quote

*The most important and productive approach to learning is for the* student *to rediscover and recreate anew the answers and methods of the past. Thus, the ideal is to present the student with a series of problems and questions and point to some of the answers that have been obtained over the past decades. The traditional method of confronting the student not with the problem but with the finished solution means depriving him or her of all excitement, to shut off the creative impulse, to reduce the adventure of humankind to a dusty heap of theorems. The issue, then, is to present some of the unanswered and important problems which we continue to confront. For it may be asserted that what we have truly learned and understood we discovered ourselves.*

Dorf, R.: *Modern Control Systems*, Addison-Wesley Publishing Co., Sydney, 1980





**Example – Precision Peak Detector**

Consider the following circuit:

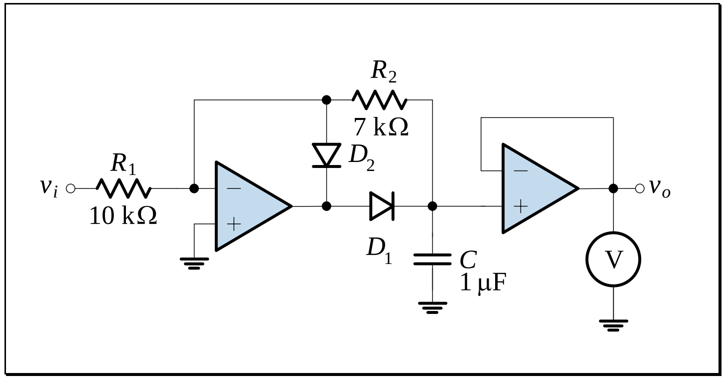



**Figure 12B.1**

The meter is a DC moving coil voltmeter that responds to the *average* of $v_o$.

The input is $v_i = \cos(2000\pi t)\text{ V}$.

We want to sketch $v_i$ and $v_o$, explain how the circuit works, and determine the meter reading.

The second op-amp circuit is just a buffer that prevents the voltmeter from loading the first circuit. Therefore an analysis of the capacitor voltage will give us $v_o$ directly.

First, consider the input sinusoid in a positive half-cycle, with the capacitor initially uncharged:

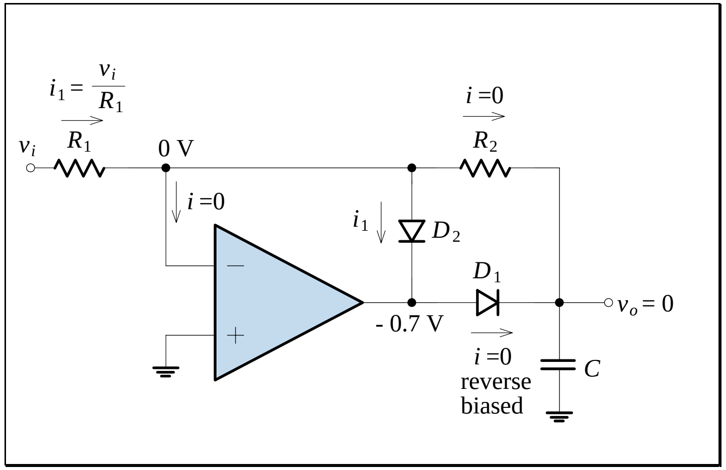



**Figure 12B.2**

The output is 0 V.





Next consider the first half of a negative half-cycle (up until the input's negative peak):

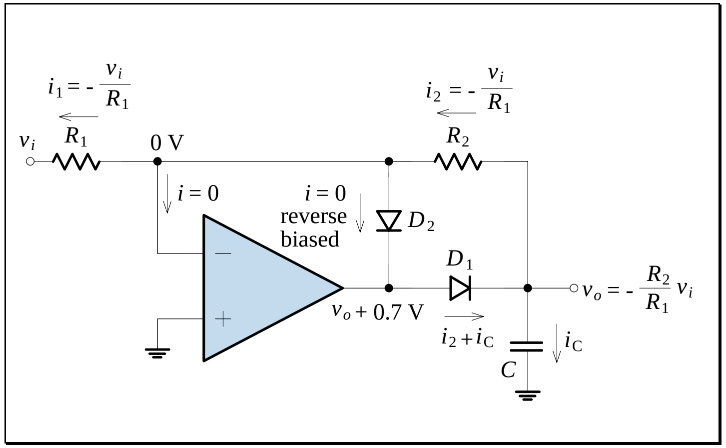



**Figure 12B.3**

The output is therefore $v_o = -0.7v_i$, which is a positive sinusoid, with a peak of 0.7 V. After the input has just passed it's negative peak, the output voltage should follow according to $v_o = -0.7v_i$. But the output voltage is across a capacitor, and for the capacitor voltage to decrease, we must remove charge from it. This means current in the upwards direction in Figure 12B.3. This current cannot go through diode $D_1$, so the capacitor must discharge through $R_2$, which will limit the rate of discharge.

We now have a conflicting requirement at the output – the capacitor must discharge according to the usual natural transient response of a charged capacitor in series with a resistor, but the op-amp would like to maintain the output voltage as a sinusoid.

The op-amp circuit must therefore change state!

Let's assume that diode $D_1$ turns off, and diode $D_2$ turns on (and therefore the op-amp will have a negative feedback path which will maintain a virtual short-circuit at the op-amp input). Then since the left side of $R_2$ is at 0 V, then we know from the usual transient analysis that the capacitor voltage is a decaying exponential given by:

$$\begin{aligned} v_o &= v_o(0)e^{-t/R_2C} \\ &= 0.7e^{-143t} \\ &\approx 0.7(1-143t) \end{aligned} \tag{12B.1}$$

where the last approximation can be made because the time constant $\tau = R_2C$ is much larger than the period of the input signal, and we have assumed $t = 0$ is the instant when the capacitor starts discharging.

The currents under this discharging condition are shown below:

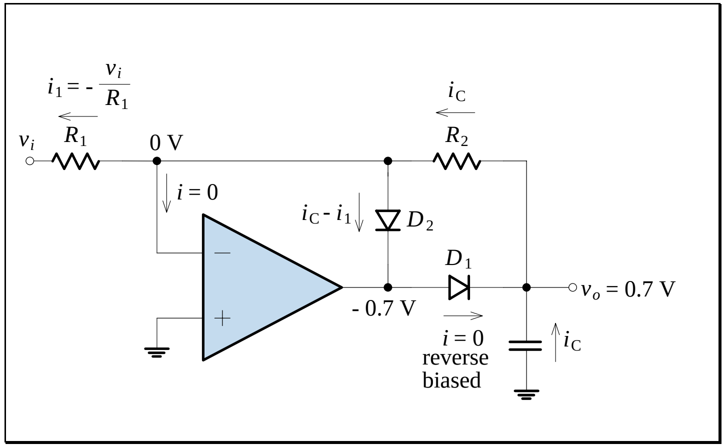



**Figure 12B.4**





The output voltage remains at approximately 0.7 V because the discharge rate is very slow. Since the current $i_C \approx 0.7/R_2 = 0.1\,\text{mA}$ is greater than $i_1 = -v_i/R_1$, the excess current must pass through diode $D_2$ and into the op-amp. The op-amp's output terminal is therefore at approximately -0.7 V, and so $D_1$ is indeed reverse-biased.

This state of the circuit must exist until the input once again makes $v_o = -0.7v_i$ greater than the capacitor voltage (which will occur near the next negative peak of the input waveform). Then the circuit will charge the capacitor back up to the 0.7 times the input peak.

Note that in the above analysis, it was always assumed that there was a negative feedback path around the op-amp, and that a virtual short-circuit existed between the op-amp's input terminals.

The input and output waveforms are shown below ( the decay of $v_o$ has been exaggerated for clarity):

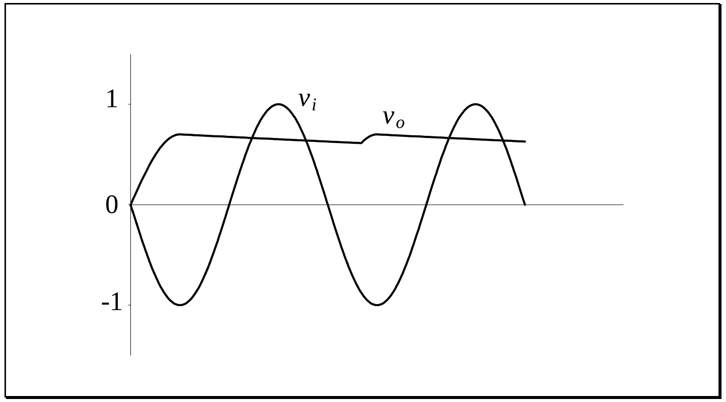



**Figure 12B.5**

The DC moving coil meter will respond to the average of the output voltage, and therefore read approximately 0.7 V (which is roughly equal to the RMS value of the input sinusoid).

**Example – Amplifiers, Integrator**

The op-amps in the circuits below have very high gain and input resistance. We want to determine the output voltage and input resistance of each circuit, if $v_i = \cos(2000\pi t)$ V.

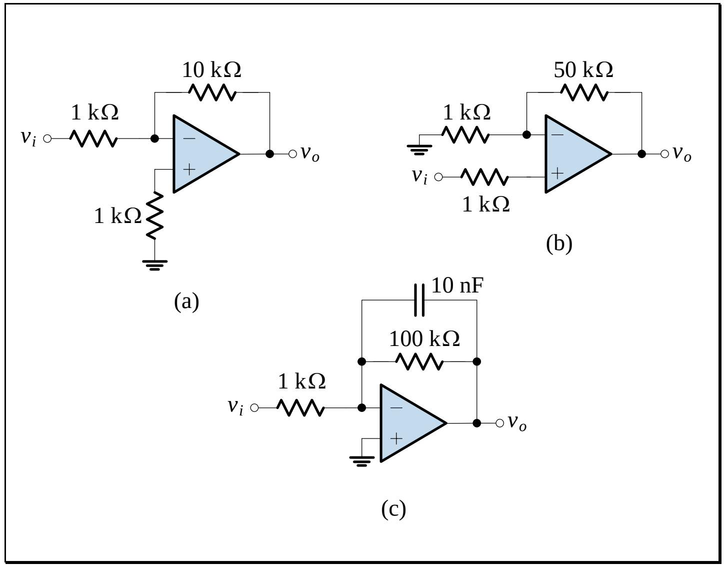



**Figure 12B.6**

(a) The resistor connected to the non-inverting terminal has no effect on the operation of the circuit, since the input resistance of the op-amp is very high and there is therefore no current. It would normally be put into a real circuit to counter the effects of bias currents. The circuit is therefore just an inverting amplifier, with a gain of –10. The output is then $v_o = -10\cos(2000\pi t)$. The input resistance, by definition, is the input voltage divided by the input current. Since the right-hand side of the $1\,\text{k}\Omega$ is at a virtual common, the input current is $v_i/1\,\text{k}$. The input resistance is therefore $1\,\text{k}\Omega$.





(b) Again, the resistor attached to the non-inverting terminal has no current in it, thanks to the op-amps very high input resistance. The circuit is then just a non-inverting amplifier, with a gain of $1 + 50/1 = 51$. The output is then $v_o = 51\cos(2000\pi t)$. The input resistance is ideally infinite.

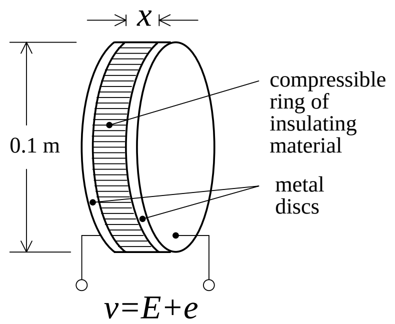
(c) The circuit is an integrator. The reactance of the capacitor at the input's frequency is $X = 1/2\pi fC = 15915\,\Omega$. Combining this with the resistor in parallel with it, we get an overall feedback impedance of $\mathbf{Z}_2 = 2470 - j15522 = 15717\angle -81°$. Then we use the generalized form for the output of an inverting amplifier, $\mathbf{V}_o = -\mathbf{Z}_2/\mathbf{Z}_1\mathbf{V}_i$. The input phasor is just $\mathbf{V}_i = 1\angle 0°$, so we get $\mathbf{V}_o = -\dfrac{15717\angle -81°}{1000}1\angle 0° = 15.72\angle 99°$ for the output phasor. Converting this to the time-domain, the output is $v_o = 15.72\cos(2000\pi t + 99°)$.

Note: You could perform an approximate analysis in the time-domain by ignoring the feedback resistor – in which case you get $v_o = -15.92\sin(2000\pi t)$, which is close to the exact answer.

## Problems

### 1. [Voltage excited electrostatic transducer – loudspeaker]

The construction of an electrostatic transducer is shown below:




The applied voltage $v$ produces a force on the metal discs which alters the spacing $x$ and acoustic waves result.

For an electrostatic transducer $F \propto v^2$. However, for good sound reproduction $x \propto \text{audio signal}$ is desirable. Therefore an applied voltage $v = E + e$ is used, where $E$ is a constant and $E >> e$ so that $v^2 = E^2 + 2Ee + e^2 \approx E(E + 2e)$.

(a) If the relative permittivity of the insulating ring is $\varepsilon_r = 1$, the DC voltage is $E = 1\,\text{kV}$, and the spacing between plates is $x \approx 0.5\,\text{mm}$, show that:

$$F \approx 0.139 + 2.78\times10^{-4} e \text{ N}$$

(b) The insulating ring has spring constant $K_r = 300\,\text{Nm}^{-1}$ and $e = 100\sin(\omega t)$. Determine the peak to peak oscillation in $x$.





**2.**

A toroidal, iron-cored ($\mu_r = 2000$) coil has 1000 uniformly distributed turns. The core has a mean length of 200 mm and an airgap of length 10 mm. The coil carries an AC current $i = 2\sin(100\pi t)$.

(a) Determine the peak values of the magnetic flux density and magnetic field intensity in the gap.

(b) A second coil with 2000 turns is wound over the first coil and its terminals connected to a $100\,\Omega$ resistor. The flux density in the core equals the value calculated in (a). Determine the supply current and voltage if the peak value of the load current is 2 A.

(c) What is the mutual inductance of the system?

Ignore fringing and leakage flux, and assume perfect magnetic coupling between the coils.

# Greek Alphabet

| Name | Capital | Small | Commonly Used to Designate |
|---|---|---|---|
| Alpha | Α | α | Angles, coefficients, attenuation |
| Beta | Β | β | Angles, coefficients, phase constant |
| Gamma | Γ | γ | Complex propagation constant (cap), electrical conductivity, propagation constant |
| Delta | Δ | δ | Increment or decrement (cap or small), density, angles |
| Epsilon | Ε | ε | Permittivity |
| Zeta | Ζ | ζ | Coordinates, coefficients |
| Eta | Η | η | Efficiency, coordinates |
| Theta | Θ | θ | Angular phase displacement |
| Iota | Ι | ι | |
| Kappa | Κ | κ | Susceptibility, coupling coefficient |
| Lambda | Λ | λ | Permeance (cap), wavelength |
| Mu | Μ | μ | Permeability, prefix micro |
| Nu | Ν | ν | Frequency |
| Xi | Ξ | ξ | Coordinates |
| Omicron | Ο | ο | |
| Pi | Π | π | Product (cap), 3.14159265… |
| Rho | Ρ | ρ | Resistivity, volume charge density, coordinates |
| Sigma | Σ | σ | Summation (cap), surface charge density, electrical conductivity |
| Tau | Τ | τ | Time constant |
| Upsilon | Υ | υ | |
| Phi | Φ | φ | Magnetic flux (cap), scalar potential, angles |
| Chi | Χ | χ | Electric susceptibility, angles |
| Psi | Ψ | ψ | Electric flux (cap), coordinates |
| Omega | Ω | ω | Resistance in ohms (cap), solid angle (cap), angular frequency |

Note: Small letter is used except where capital (cap) is indicated.

# Answers

### 1A.1

(a) $\mathbf{E} = \dfrac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$

(b) $\mathbf{E} = \mathbf{0}$

(c) $\mathbf{E} = \dfrac{\lambda}{2\pi\varepsilon_0 r}\hat{\mathbf{r}}$

(d) $\mathbf{E} = \dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{r}}$

### 1A.2

(a) $V_{ba} = \dfrac{q}{4\pi\varepsilon_0}\left(\dfrac{1}{r_b} - \dfrac{1}{r_a}\right)$

(b) $V_{ba} = \dfrac{q}{4\pi\varepsilon_0}\left(\dfrac{1}{r_b} - \dfrac{1}{r_a}\right)$ if $r_a > a, \quad r_b > a$

$V_{ba} = \dfrac{q}{4\pi\varepsilon_0}\left(\dfrac{1}{a} - \dfrac{1}{r_a}\right)$ if $r_a > a, \quad r_b < a$

$V_{ba} = 0$ if $r_a < a, \quad r_b < a$

(c) $V_{ba} = \dfrac{\lambda}{2\pi\varepsilon_0}\ln\left(\dfrac{r_a}{r_b}\right)$

(d) $V_{ba} = \dfrac{\sigma}{2\varepsilon_0}\left(r_a - r_b\right)$

### 1A.3

$$\frac{C}{l} = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(d_2/d_1\right)}$$

**1A.4**

$$\mathbf{F} = \frac{-eQ}{4\pi\varepsilon_0 d^2}\hat{\mathbf{r}}, \qquad \hat{\mathbf{r}} = \text{unit vector pointing to electron}$$

**1A.5**

**1A.6**

$27.3\ \mu\text{A}$

**1A.7**

(b) $63.7\ \Omega$ (c) $0.796\ \text{V}$

**1A.9**

$$C = \frac{4\pi\varepsilon_0 R_1 R_2}{R_1 - R_2}, \quad R_1 > R_2$$

**1B.1**

(a) $6.28\times 10^{-5}\ \text{T}$ (b) $6.28\times 10^{-5}\ \text{T}$ (c) $0.628\ \text{T}$

**1B.3**

$$B = \mu_0 \frac{i}{R}\left(\frac{1}{2\pi} + \frac{1}{4}\right) = 5.2\times 10^{-4}\ \text{T}$$

**1B.4**

$1.39\times 10^{-2}\ \text{T}$

**1B.5**

(a) *G* yes, *G'* no (b) *G* no, *G'* yes (c) *G* yes, *G'* yes

**1B.6**

(b) $e = \frac{1}{2}\omega B r_0^2\ \text{V}$

**1B.7**

960 N





**1B.8**

$2\times 10^{-4}\ln 2\ \text{Wb/km}$

**1B.9**

$4.33\times 10^{-2}\ \text{N}$

**1B.10**

(a) $\lambda = \frac{2bN}{\alpha}\sin\frac{\alpha a}{2}\hat{B}\sin\alpha X$ (b) $e = -2bN\sin\frac{\alpha a}{2}\hat{B}v\cos\alpha X$

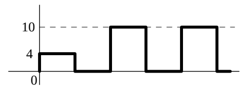
(c) $a = \pi/\alpha$ or an odd multiple

(d) $e = -2bN\sin\frac{\alpha a}{2}\left(\frac{\omega}{\alpha} - v\right)\hat{B}\cos(\omega t - \alpha X), \quad e = 0$

**4A.2**

(i) $v_o = \frac{R_L}{R_L + R_S/2}(V - e_{fd})$ (ii) $v_o = \frac{R_L}{R_L + R_S}(V - e_{fd})$

**4A.3**




**4A.4**

(a)

(i) 3 V, 1.3 mA (ii) $-5\frac{15}{47}$ V, $\frac{2}{47}$ mA

(b)

(i) 2.3 V, 1.23 mA (ii) $-5\frac{187}{235}$ V, $\frac{3}{235}$ mA

**4B.1**

(a) 1.33 T, 1200 A (b) 1.66 T, 1400 A

**4B.2**

(a) $5.03 \times 10^5$ H$^{-1}$, 0.5 T (b) $1.52 \times 10^5$ H$^{-1}$, 0.39 T

**4B.3**

(a) 1.25 A DC $(\rightarrow)$ (b) $L_{12} = L_{21} = 1.\dot{7}\ \text{H}$,

**4B.4**

(a) $B_m = 1.25\ \text{T}$

**4B.5**

(a) $x_{\max} = 1.81\ \text{mm}$

**5B.1**

(a) 1.4 T, 0.5 A 56 H; 0.55 T, 39 H

(b) -780 V

(c) 6.284 kV, 0.25 A, 1 T

**5B.4**

(a) 1.61 H (b) 0.236 H, $\text{T} = 13.7\ \text{ms}$

**5B.6**

Battery is better by a factor of 1000.





**6A.1**

(i) 35 mA, 3.75 kΩ (ii) 162.5 V, 293.8 V

**6B.1**

1.212, 115 V / 26.4 A, 95 V / 31.6 A

**6B.2**

6.91

**6B.3**

(b) $L_{21} \approx 0.5\ \text{H}$, $L_{31} \approx 0.25\ \text{H}$ (c) $\hat{v}_1 = 10\pi\ \text{V}$, $\hat{i}_1 \approx 0.2\ \text{A}$

**6B.4**

(b) $i_S = 0$ (c) $\hat{v}_1 + \hat{v}_3 = 15\ \text{V}$, $\hat{v}_2 = 15\ \text{V}$, $\hat{v}_4 = \hat{v}_1$, $\hat{v}_5 = \hat{v}_2$, $\hat{v}_6 = \hat{v}_3$

**8B.1**

(a) 2.685 mm (b) No

**8B.2**

(b) 4.189 mm, 5730 N (c) 578.3 kg (d) 202.5 mA (e) 11.72 ms$^{-2}$ up

**8B.3**

(a) 0.255 A **<u>Hint</u>**: Flux is a maximum when plunger is fully in.

**8B.4**

(a) $x \le 3.53\ \text{mm}$ (b) 8.9 J (c) 1.9 J

**8B.5**

885 μJ

**8B.7**

(b) $F = 55\ \text{N}$

**10A.1**

$1\,\text{M}\Omega$, $1\,\text{k}\Omega$, 80

**10A.2**

$20\,\text{k}\Omega$, 3 mS

**10A.3**

$R_o \geq 99R_L$, $A_{is}i_i = \frac{A_{vo}v_i}{R_o}$

**10A.4**

63.8, 90, 63.1, 1 Hz to 1 MHz

**10A.5**

26.7, 24.3

**10B.2**

$L_x = R_2R_3C_1$ and $R_x = \frac{R_2R_3}{R_1}$

**10B.3**

$L_x = \frac{R_2R_3C_1}{1+\omega^2R_1^2C_1^2}$ and $R_x = \frac{\omega^2R_1C_1^2R_2R_3}{1+\omega^2R_1^2C_1^2}$

**12B.1**

(b) 0.185 mm

**12B.2**

(a) 0.25 T, $199\,\text{kAm}^{-1}$ (b) $\sqrt{20}$ A, 100 V (c) $1/\pi$ H