3 Nodal Analysis

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Introduction

After becoming familiar with Ohm's Law and Kirchhoff's Laws and their application in the analysis of simple series and parallel resistive circuits, we must begin to analyse more complicated and practical circuits.

Physical systems that we want to analyse and design include electronic control circuits, communication systems, energy converters such as motors and generators, power distribution systems, mobile devices and embedded systems. We will also be confronted with allied problems involving heat flow, fluid flow, and the behaviour of various mechanical systems.

To cope with large and complex circuits, we need powerful and general methods of circuit analysis. Nodal analysis is a method which can be applied to any circuit. This method is widely used in hand design and computer simulation.

3.1 Nodal Analysis

In general terms, nodal analysis for a circuit with N nodes proceeds as follows:

1. Select one node as the reference node, or *common* (all nodal voltages are defined with respect to this node in a positive sense).

2. Assign a voltage to each of the remaining $(N-1)$ nodes.

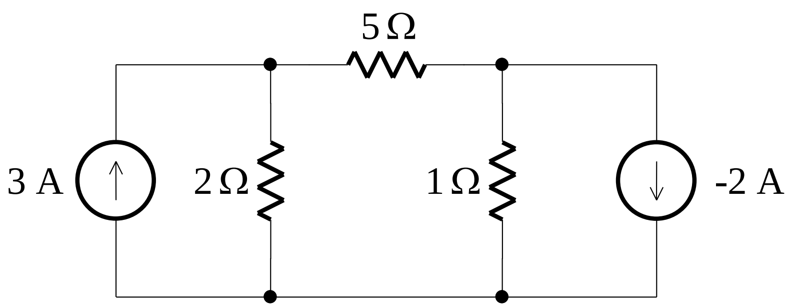
3. Write KCL at each node, in terms of the nodal voltages.

4. Solve the resulting set of simultaneous equations.

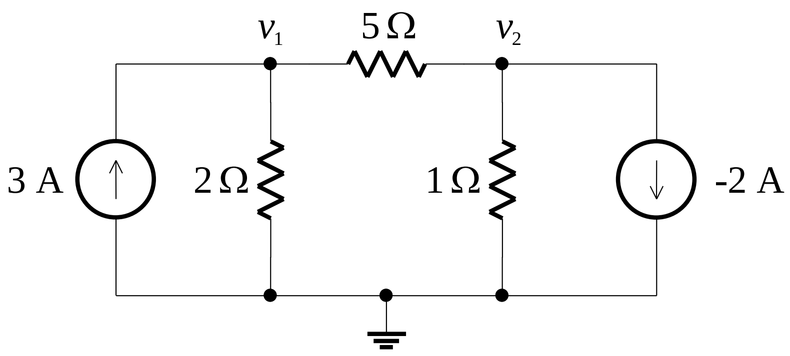
The general principle of nodal analysis

EXAMPLE 3.1 Nodal Analysis with Independent Sources

We apply nodal analysis to the following 3-node circuit:



Following the steps above, we assign a reference node and then assign nodal voltages:



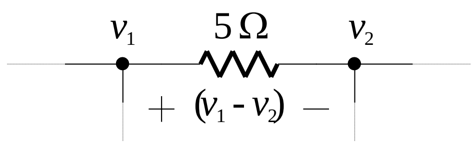
We chose the bottom node as the reference node, but either of the other two nodes could have been selected. A little simplification in the resultant equations is obtained if the node to which the greatest number of branches is connected is identified as the reference node.

In many practical circuits the reference node is one end of a power supply which is generally connected to a metallic case or chassis in which the circuit resides; the chassis is often connected through a good conductor to the Earth. Thus, the metallic case may be called “ground”, or “earth”, and this node becomes the most convenient reference node.

The distinction between “common” and “earth”

To avoid confusion, the reference node will be called the “common” unless it has been specifically connected to the Earth (such as the outside conductor on a digital storage oscilloscope, function generator, etc.).

Note that the voltage across any branch in a circuit may be expressed in terms of nodal voltages. For example, in our circuit the voltage across the $5\,\Omega$ resistor is $(v_1 - v_2)$ with the positive polarity reference on the left:



We must now apply KCL to nodes 1 and 2. We do this be equating the total current *leaving* a node to zero. Thus:

$$\frac{v_1}{2} + \frac{v_1 - v_2}{5} - 3 = 0$$
$$\frac{v_2 - v_1}{5} + \frac{v_2}{1} + (-2) = 0$$

Simplifying, the equations can be written:

$$0.7v_1 - 0.2v_2 = 3$$
$$-0.2v_1 + 1.2v_2 = 2$$

Rewriting in matrix notation, we have:

$$\begin{bmatrix} 0.7 & -0.2 \\ -0.2 & 1.2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

These equations may be solved by a simple process of elimination of variables, or by Cramer's rule and determinants. Using the latter method we have:

$$v_1 = \frac{\begin{vmatrix} 3 & -0.2 \\ 2 & 1.2 \end{vmatrix}}{\begin{vmatrix} 0.7 & -0.2 \\ -0.2 & 1.2 \end{vmatrix}} = \frac{3.6+0.4}{0.84-0.04} = \frac{4}{0.8} = 5\text{ V}$$

$$v_2 = \frac{\begin{vmatrix} 0.7 & 3 \\ -0.2 & 2 \end{vmatrix}}{0.8} = \frac{1.4+0.6}{0.8} = \frac{2}{0.8} = 2.5\text{ V}$$

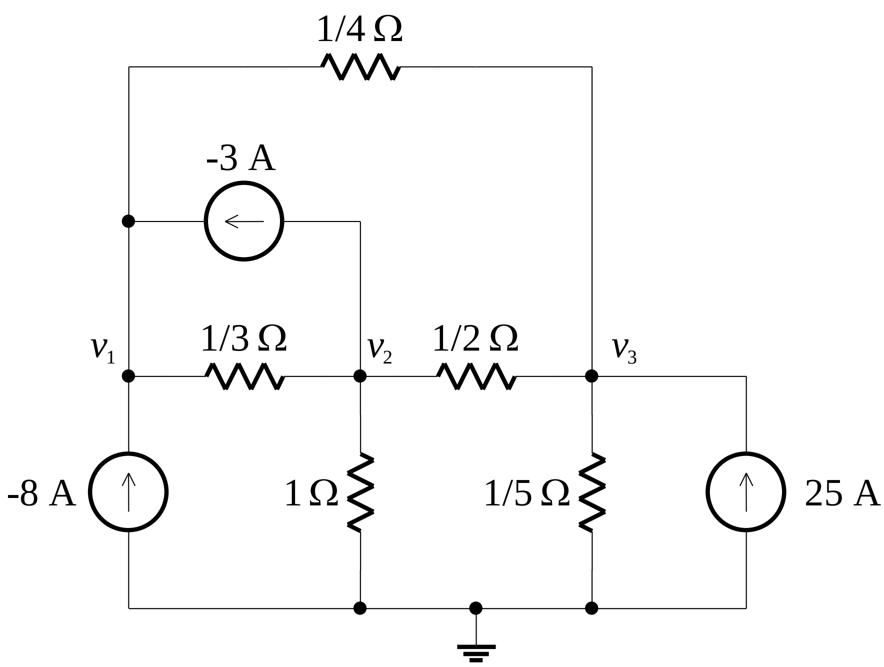
Everything is now known about the circuit – any voltage, current or power in the circuit may be found in one step. For example, the voltage at node 1 with respect to node 2 is $(v_1 - v_2) = 2.5\text{ V}$, and the current directed downward through the $2\,\Omega$ resistor is $v_1/2 = 2.5\text{ A}$.

3.1.1 Circuits with Resistors and Independent Current Sources Only

A further example will reveal some interesting mathematical features of nodal analysis, at least for the case of circuits containing only resistors and independent current sources. We will find it is much easier to consider conductance, *G*, rather than resistance, *R*, in the formulation of the equations.

EXAMPLE 3.2 Nodal Analysis with Independent Sources Only

A circuit is shown below with a convenient reference node and nodal voltages specified.



We sum the currents leaving node 1:

$$3(v_1 - v_2) + 4(v_1 - v_3) - (-8) - (-3) = 0$$
$$7v_1 - 3v_2 - 4v_3 = -11$$

At node 2:

$$3(v_2 - v_1) + 1v_2 + 2(v_2 - v_3) - 3 = 0$$
$$-3v_1 + 6v_2 - 2v_3 = 3$$

At node 3:

$$4(v_3 - v_1) + 2(v_3 - v_2) + 5v_3 - 25 = 0$$
$$-4v_1 - 2v_2 + 11v_3 = 25$$

Rewriting in matrix notation, we have:

$$\begin{bmatrix} 7 & -3 & -4 \\ -3 & 6 & -2 \\ -4 & -2 & 11 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -11 \\ 3 \\ 25 \end{bmatrix}$$

For circuits that contain only resistors and independent current sources, we define the *conductance matrix* of the circuit as:

$$\mathbf{G} = \begin{bmatrix} 7 & -3 & -4 \\ -3 & 6 & -2 \\ -4 & -2 & 11 \end{bmatrix}$$

It should be noted that the nine elements of the matrix are the ordered array of the coefficients of the KCL equations, each of which is a conductance value. Thr first row is composed of the coefficients of the Kirchhoff current law equation at the first node, the coefficients being given in the order of v_1, v_2 and v_3. The second row applies to the second node, and so on.

The conductance matrix defined

The major diagonal (upper left to lower right) has elements that are positive. The conductance matrix is symmetrical about the major diagonal, and all elements not on this diagonal are negative. This is a general consequence of the systematic way in which we ordered the equations, and in circuits consisting of only resistors and independent current sources it provides a check against errors committed in writing the circuit equations.

We also define the voltage and current source vectors as:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \mathbf{i} = \begin{bmatrix} -11 \\ 3 \\ 25 \end{bmatrix}$$

Nodal analysis expressed in matrix notation

Our KCL equations can therefore be written succinctly in matrix notation as:

$$\mathbf{Gv} = \mathbf{i}$$

The solution of the matrix equation is just:

$$\mathbf{v} = \mathbf{G}^{-1}\mathbf{i}$$

Computer programs that do nodal analysis use sophisticated numerical methods to efficiently invert the $\mathbf{G}$ matrix and solve for $\mathbf{v}$. When solving the equations by hand we resort to matrix reduction techniques, or use Cramer's rule (up to 3 x 3). Thus:

$$v_1 = \frac{\begin{vmatrix} -11 & -3 & -4 \\ 3 & 6 & -2 \\ 25 & -2 & 11 \end{vmatrix}}{\begin{vmatrix} 7 & -3 & -4 \\ -3 & 6 & -2 \\ -4 & -2 & 11 \end{vmatrix}}$$

To reduce work, we expand the numerator and denominator determinants by minors along their first columns to get:

$$\begin{aligned} v_1 &= \frac{-11\begin{vmatrix} 6 & -2 \\ -2 & 11 \end{vmatrix} - 3\begin{vmatrix} -3 & -4 \\ -2 & 11 \end{vmatrix} + 25\begin{vmatrix} -3 & -4 \\ 6 & -2 \end{vmatrix}}{7\begin{vmatrix} 6 & -2 \\ -2 & 11 \end{vmatrix} - (-3)\begin{vmatrix} -3 & -4 \\ -2 & 11 \end{vmatrix} + (-4)\begin{vmatrix} -3 & -4 \\ 6 & -2 \end{vmatrix}} \\ &= \frac{-11(62) - 3(-41) + 25(30)}{7(62) + 3(-41) - 4(30)} = \frac{-682 + 123 + 750}{434 - 123 - 120} \\ &= \frac{191}{191} = 1\text{ V} \end{aligned}$$

Similarly:

$$v_2 = \frac{\begin{vmatrix} 7 & -11 & -4 \\ -3 & 3 & -2 \\ -4 & 25 & 11 \end{vmatrix}}{|191|} = 2\text{ V} \qquad v_3 = \frac{\begin{vmatrix} 7 & -3 & -11 \\ -3 & 6 & 3 \\ -4 & -2 & 25 \end{vmatrix}}{|191|} = 3\text{ V}$$

3.1.2 Nodal Analysis Using Branch Element Stamps

The previous example shows that nodal analysis leads to the equation $\mathbf{Gv} = \mathbf{i}$. We will now develop a method whereby the equation $\mathbf{Gv} = \mathbf{i}$ can be built up on an element-by-element basis by inspection of each branch in the circuit.

Consider a resistive element connected between nodes *i* and *j*:

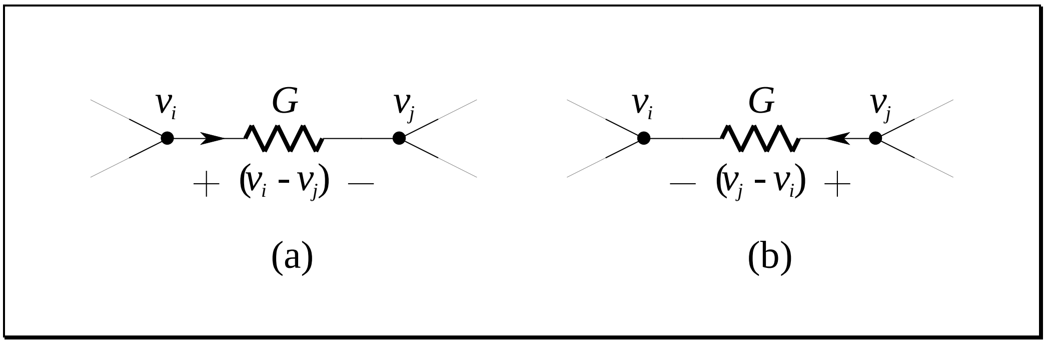


Figure 3.1

Suppose that we are writing the i^{th} KCL equation because we are considering the current leaving node *i* (see Figure 3.1a). The term that we would write in this equation to take into account the branch connecting nodes *i* and *j* is:

$$\cdots + G(v_i - v_j) + \cdots = 0 \tag{3.1}$$

This term appears in the i^{th} row when writing out the matrix equation.

If we are dealing with the j^{th} KCL equation because we are considering the current leaving node *j* (see Figure 3.1b) then the term that we would write in this equation to take into account the branch connecting nodes *j* and *i* is:

$$\cdots + G(v_j - v_i) + \cdots = 0 \tag{3.2}$$

This term appears in the j^{th} row when writing out the matrix equation.

Thus, the branch between nodes i and j contributes the following *element stamp* to the conductance matrix, $\mathbf{G}$:

The element stamp for a conductance

$$\begin{array}{c c} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{bmatrix} G & -G \\ -G & G \end{bmatrix} \end{array} \tag{3.3}$$

If node i or node j is the reference node, then the corresponding row and column are eliminated from the element stamp shown above.

For any circuit containing only resistors and independent current sources, the conductance matrix can now be built up by inspection. The result will be a $\mathbf{G}$ matrix where each diagonal element g_{ii} is the sum of conductances connected to node i, and each off-diagonal element g_{ij} is the total conductance between nodes i and j but with a negative sign.

Now consider a current source connected between nodes i and j:

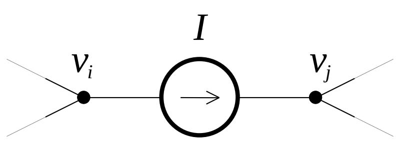


Figure 3.2

In writing out the i^{th} KCL equation we would introduce the term:

$$\cdots + I + \cdots = 0 \tag{3.4}$$

In writing out the j^{th} KCL equation we would introduce the term:

$$\cdots - I + \cdots = 0 \tag{3.5}$$

Thus, a current source contributes to the right-hand side (rhs) of the matrix equation the terms:

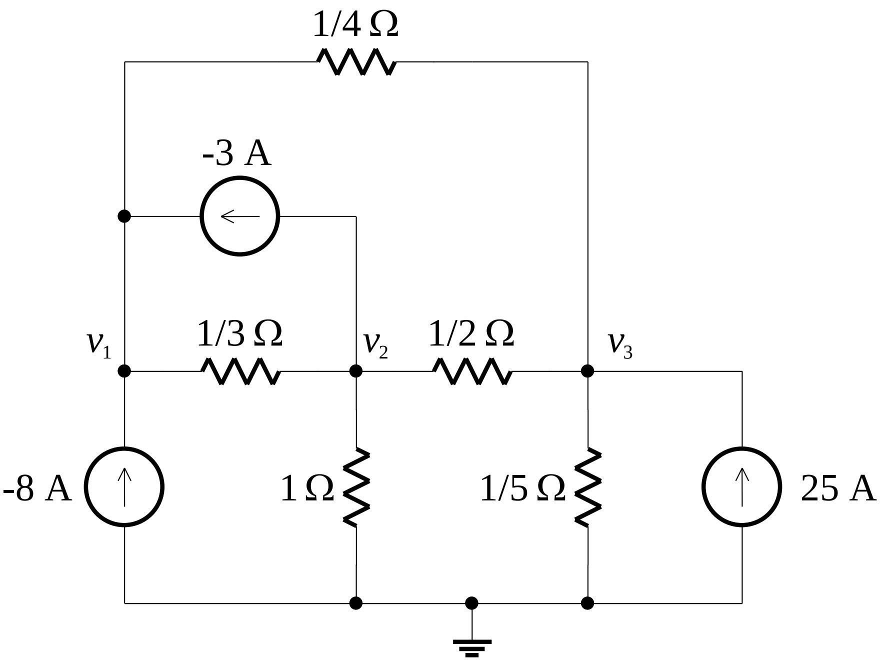
$$\begin{array}{c} i \\ j \end{array}\begin{bmatrix} -I \\ I \end{bmatrix} \tag{3.6}$$

The element stamp for an independent current source

Thus, the $\mathbf{i}$ vector can also be built up by inspection – each row is the addition of all current sources *entering* a particular node. This makes sense since $\mathbf{Gv} = \mathbf{i}$ is the mathematical expression for KCL in the form of "current leaving a node $=$ current entering a node".

EXAMPLE 3.3 Nodal Analysis Using the "Formal" Approach

We will analyse the previous circuit but use the "formal" approach to nodal analysis.



By inspection of each branch, we build the matrix equation:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3+4 & -3 & -4 \\ -3 & 1+3+2 & -2 \\ -4 & -2 & 5+4+2 \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \end{array} \begin{array}{c} \\ = \end{array} \begin{array}{c} \\ \begin{bmatrix} (-8)+(-3) \\ -(-3) \\ 25 \end{bmatrix} \end{array}$$

This is the same equation as derived previously.

3.1.3 Circuits with Voltage Sources

Voltage sources present a problem in undertaking nodal analysis, since by definition the voltage across a voltage source is *independent* of the current through it. Thus, when we consider a branch with a voltage source when writing a nodal equation, there is no way by which we can express the current through the branch as a function of the nodal voltages across the branch.

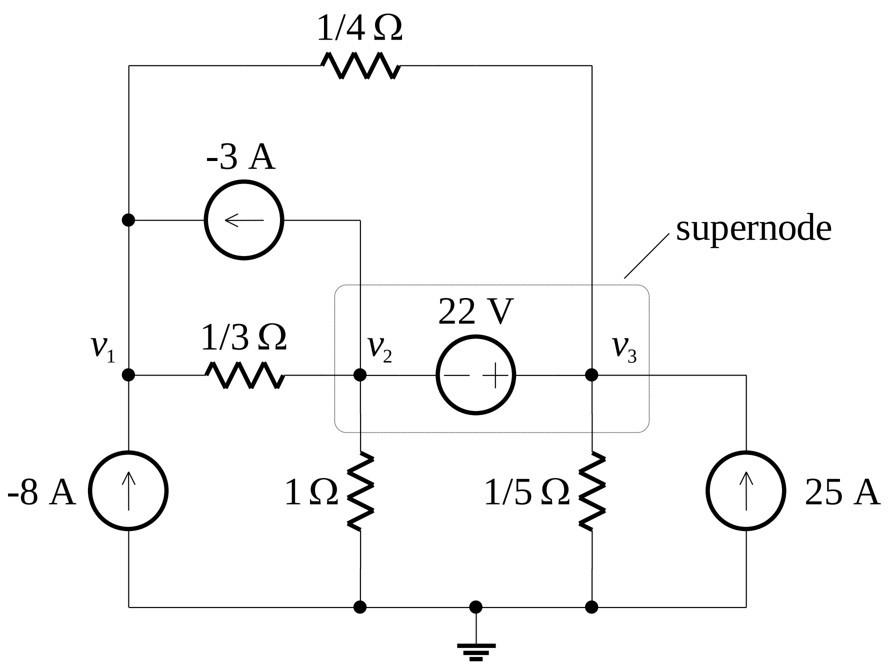
There are two ways around this problem. The more difficult is to assign an unknown current to each branch with a voltage source, proceed to apply KCL at each node, and then apply KVL across each branch with a voltage source. The result is a set of equations with an increased number of unknown variables.

The concept of a supernode

The easier method is to introduce the concept of a *supernode*. A supernode encapsulates the voltage source, and we apply KCL to both end nodes at the same time. The result is that the number of nodes at which we must apply KCL is reduced by the number of voltage sources in the circuit.

EXAMPLE 3.4 Nodal Analysis with Voltage Sources

Consider the circuit shown below, which is the same as the previous circuit except the $1/2\,\Omega$ resistor between nodes 2 and 3 has been replaced by a 22 V voltage source:



KCL at node 1 remains unchanged:

$$3(v_1 - v_2) + 4(v_1 - v_3) - (-8) - (-3) = 0$$
$$7v_1 - 3v_2 - 4v_3 = -11$$

We find six branches connected to the supernode around the 22 V source (suggested by a broken line in the figure). Beginning with the $1/3\,\Omega$ resistor branch and working clockwise, we sum the six currents leaving this supernode:

$$3(v_2 - v_1) + (-3) + 4(v_3 - v_1) + (-25) + 5v_3 + 1v_2 = 0$$
$$-7v_1 + 4v_2 + 9v_3 = 28$$

We need one additional equation since we have three unknowns, and this is provided by KVL between nodes 2 and 3 inside the supernode:

$$v_3 - v_2 = 22$$

Rewriting these last three equations in matrix form, we have:

$$\begin{bmatrix} 7 & -3 & -4 \\ -7 & 4 & 9 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -11 \\ 28 \\ 22 \end{bmatrix}$$

The solution turns out to be $v_1 = -4.5\text{ V}$, $v_2 = -15.5\text{ V}$ and $v_3 = 6.5\text{ V}$.

Note the lack of symmetry about the major diagonal in the $\mathbf{G}$ matrix as well as the fact that not all of the off-diagonal elements are negative. This is the result of the presence of the voltage source. Note also that it does not make sense to call the $\mathbf{G}$ matrix the *conductance* matrix, for the bottom row comes from the equation $-v_2 + v_3 = 22$, and this equation does not have any terms that are related to a conductance.

The presence of a dependent source destroys the symmetry in the **G** matrix

A supernode can contain any number of independent or dependent voltage sources. In general, the analysis procedure is the same as the example above – *one* KCL equation is written for currents leaving the supernode, and then a KVL equation is written for *each* voltage source inside the supernode.

3.1.4 Summary of Nodal Analysis

The general procedure to follow when undertaking nodal analysis

We perform nodal analysis for any resistive circuit with N nodes by the following method:

1. Make a neat, simple, circuit diagram. Indicate all element and source values. Each source should have its reference symbol.

2. Select one node as the reference node, or *common*. Then write the node voltages v_1, v_2, …, v_{N-1} at their respective nodes, remembering that each node voltage is understood to be measured with respect to the chosen reference.

3. If the circuit contains dependent sources, express those sources in terms of the variables v_1, v_2, …, v_{N-1}, if they are not already in that form.

4. If the circuit contains voltage sources, temporarily modify the given circuit by replacing each voltage source by a short-circuit to form *supernodes*, thus reducing the number of nodes by one for each voltage source that is present. The assigned nodal voltages should not be changed. Relate each supernode's source voltage to the nodal voltages.

5. Apply KCL at each of the nodes or supernodes. If the circuit has only resistors and independent current sources, then the equations may be built using the "element stamp" approach.

6. Solve the resulting set of simultaneous equations.

The technique of nodal analysis described here is completely general and can always be applied to any electrical circuit.

3.2 Summary

- Nodal analysis can be applied to any circuit. Apart from relating source voltages to nodal voltages, the equations of nodal analysis are formed from application of Kirchhoff's Current Law.

- In nodal analysis, a supernode is formed by short-circuiting a voltage source and treating the two ends as a single node.

3.3 References

Hayt, W. & Kemmerly, J.: *Engineering Circuit Analysis*, 3rd *Ed*., McGraw-Hill, 1984.

3.16

Exercises

1.

(a) Find the value of the determinant:

$$\Delta = \begin{vmatrix} 2 & -1 & 0 & -3 \\ -1 & 1 & 0 & -1 \\ 4 & 0 & 3 & -2 \\ -3 & 0 & 0 & 1 \end{vmatrix}$$

(b) Use Cramer's rule to find v_1, v_2 and v_3 if:

$$\begin{aligned} &2v_1 - 35 - v_2 + 3v_3 = 0 \\ &-2v_3 - 3v_2 - 4v_1 = 56 \\ &v_2 + 3v_1 - 28 - v_3 = 0 \end{aligned}$$