5 Circuit Analysis Techniques

Contents

Introduction

Many of the circuits that we analyse and design are *linear* circuits. Linear circuits possess the property that “outputs are proportional to inputs”, and that “a sum of inputs leads to a sum of corresponding outputs”. This is the principle of superposition and is a very important consequence of linearity. As will be seen later, this principle will enable us to analyse circuits with multiple sources in an easy way.

Nonlinear circuits can be analysed and designed with graphical methods or numerical methods (with a computer) – the mathematics that describe them can only be performed by hand in the simplest of cases. Examples of nonlinear circuits are those that contain diodes, transistors, and ferromagnetic material.

In reality all circuits are nonlinear, since there must be physical limits to the linear operation of devices, e.g. voltages will eventually break down across insulation, resistors will burn because they can’t dissipate heat to their surroundings, etc. Therefore, when we draw, analyse and design a linear circuit, we keep in mind that it is a *model* of the real physical circuit, and it is only valid under a defined range of operating conditions.

In modelling real physical circuit elements, we need to consider *practical* sources as opposed to *ideal* sources. A practical source gives a more realistic representation of a physical device. We will study methods whereby practical current and voltage sources may be interchanged without affecting the remainder of the circuit. Such sources will be called *equivalent* sources.

5.1 Linearity

A linear circuit is one that contains linear elements, independent sources, and linear dependent sources.

A linear circuit defined

A *linear element* is one that possesses a linear relationship between a cause and an effect. For example, when a voltage is impressed across a resistor, a current results, and the amount of current (the effect) is proportional to the voltage (the cause). This is expressed by Ohm's Law, $v = Ri$. Notice that a *linear element* means simply that if the cause is increased by some multiplicative constant K, then the effect is also increased by the same constant K.

A linear element defined

If a linear element's relationship is graphed as "cause" vs. "effect", the result is a *straight line through the origin*. For example, the resistor relationship is:

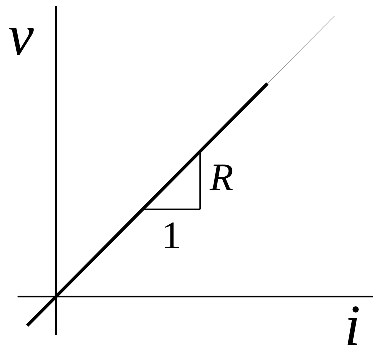


Figure 5.1

A linear relationship is defined by a straight line through the origin

A *linear dependent source* is one whose output voltage or current is proportional only to the first power of some current or voltage variable in the circuit (or a sum of such quantities). For example, a dependent voltage source given by $v_s = 0.6i_1 - 14v_2$ is linear, but $v_s = 0.6i_1^2$ and $v_s = 0.6i_1v_2$ are not.

A linear dependent source defined

From the definition of a linear circuit, it is possible to show that "the response is proportional to the source", or that multiplication of all *independent* sources by a constant K increases all the current and voltage responses by the same factor K (including the dependent source outputs).

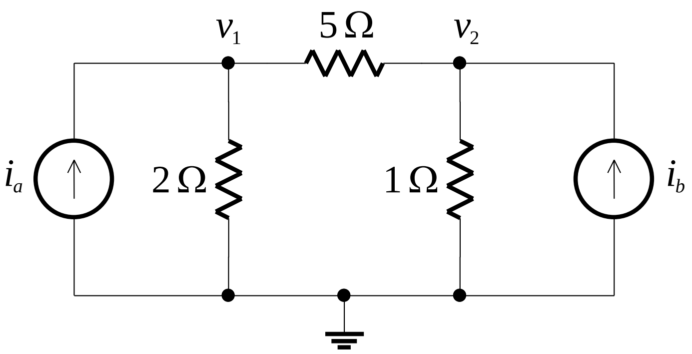
Output is proportional to input for a linear circuit

5.2 Superposition

The linearity property of a circuit leads directly to the principle of superposition. To develop the idea, consider the following example:

EXAMPLE 5.1 Superposition

We have a 3-node circuit:



There are two independent current sources which force the currents i_a and i_b into the circuit. Sources are often called *forcing functions* for this reason, and the voltages they produce at each node in this circuit may be termed *response functions*, or simply *responses*.

The two nodal equations for this circuit are:

$$\begin{aligned} 0.7v_1 - 0.2v_2 &= i_a \\ -0.2v_1 + 1.2v_2 &= i_b \end{aligned}$$

Now we perform experiment x. We change the two current sources to i_{ax} and i_{bx}; the two unknown node voltages will now be different, and we let them be v_{1x} and v_{2x}. Thus:

$$\begin{aligned} 0.7v_{1x} - 0.2v_{2x} &= i_{ax} \\ -0.2v_{1x} + 1.2v_{2x} &= i_{bx} \end{aligned}$$

If we now perform experiment *y* by changing the current sources again, we get:

$$\begin{aligned} 0.7v_{1y} - 0.2v_{2y} &= i_{ay} \\ -0.2v_{1y} + 1.2v_{2y} &= i_{by} \end{aligned}$$

We now add or *superpose* the two results of the experiments:

$$\begin{aligned} 0.7(v_{1x} + v_{1y}) - 0.2(v_{2x} + v_{2y}) &= (i_{ax} + i_{ay}) \\ -0.2(v_{1x} + v_{1y}) + 1.2(v_{2x} + v_{2y}) &= (i_{bx} + i_{by}) \end{aligned}$$

Compare this with the original set of equations:

$$\begin{aligned} 0.7v_1 - 0.2v_2 &= i_a \\ -0.2v_1 + 1.2v_2 &= i_b \end{aligned}$$

We can draw an interesting conclusion. If we let $i_{ax} + i_{ay} = i_a$, $i_{bx} + i_{by} = i_b$, then the desired responses are given by $v_1 = v_{1x} + v_{1y}$ and $v_2 = v_{2x} + v_{2y}$. That is, we may perform experiment *x* and note the responses, perform experiment *y* and note the responses, and finally add the corresponding responses. These are the responses of the original circuit to independent sources which are the sums of the independent sources used in experiments *x* and *y*.

This is the fundamental concept involved in the superposition principle. It is evident that we may break an independent source into as many pieces as we wish, so long as the algebraic sum of the pieces is equal to the original source.

Superposition allows us to treat inputs separately, then combine individual responses to obtain the total response

In practical applications of the superposition principle, we usually set each independent source to zero, so that we can analyse the circuit one source at a time.

5.2.1 Superposition Theorem

We can now state the superposition theorem as it is mostly applied to circuits:

The superposition theorem

> In any linear network containing several sources, we can calculate any response by adding algebraically all the individual responses caused by each independent source acting alone, with all other independent sources set to zero. (5.1)

When we set the value of an independent voltage source to zero, we create a *short-circuit* by definition. When we set the value of an independent current source to zero, we create an *open-circuit* by definition.

Setting a voltage source to zero creates a short-circuit. Setting a current source to zero creates an open-circuit.

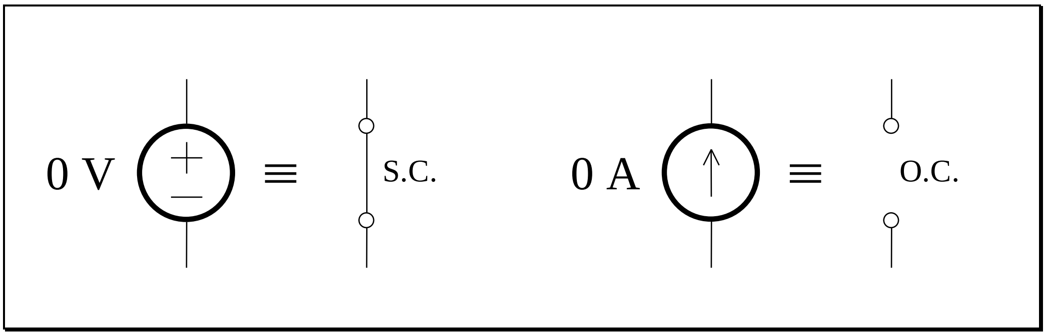


Figure 5.2

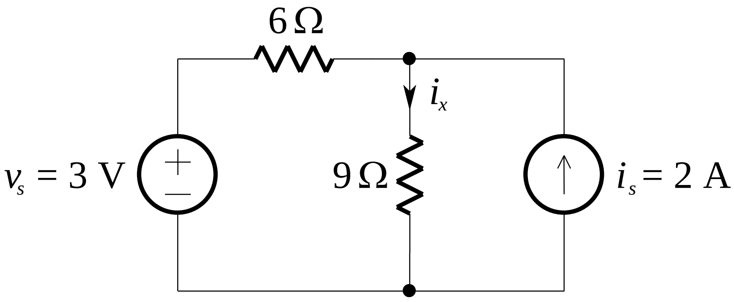
Note that *dependent* sources **cannot** be arbitrarily set to zero, and are generally active when considering *every* individual independent source.

The theorem as stated above can be made much stronger – a group of independent sources may be made active and inactive collectively. For example, sometimes it is handy to consider all voltage sources together, so that mesh analysis can be applied easily, and then all current sources together so that nodal analysis may be applied easily.

There is also no reason that an independent source must assume only its given value or zero – it is only necessary that the sum of the several values be equal to the original value. However, an inactive source almost always leads to the simplest circuit.

EXAMPLE 5.2 Superposition with Independent Sources

We use superposition in the following circuit to write an expression for the unknown branch current i_x.



We first set the current source equal to zero (an open-circuit) and obtain the portion of i_x due to the voltage source as 0.2 A. Then if we let the voltage source be zero (a short-circuit) and apply the current divider rule, the remaining portion of i_x is seen to be 0.8 A.

We may write the answer in detail as:

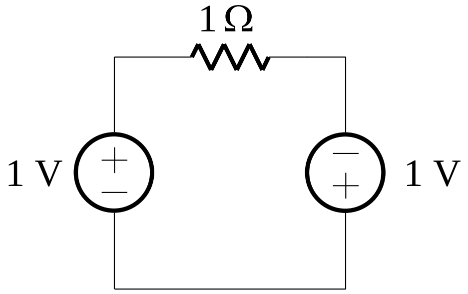
$$i_x = i_x\big|_{i_s=0} + i_x\big|_{v_s=0} = \frac{3}{6+9} + \frac{6}{6+9}2 = 0.2 + 0.8 = 1\,\mathrm{A}$$

Superposition is often *misapplied* to power in a circuit element

We must be aware of the limitations of superposition. It is applicable only to linear responses, and thus the most common nonlinear response – power – is not subject to superposition.

EXAMPLE 5.3 Superposition Cannot be Applied to Power

The circuit below contains two 1 V batteries in series.



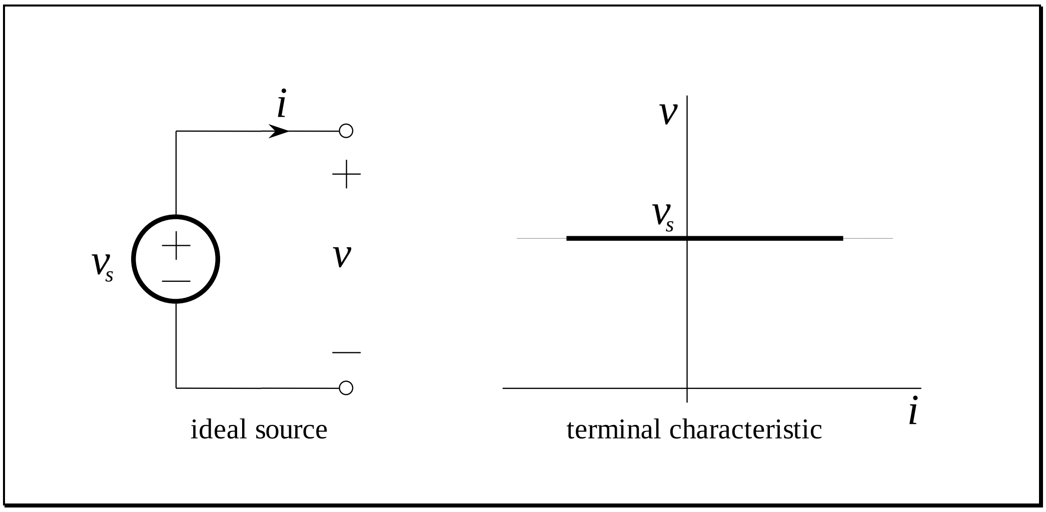
If we apply superposition, then each voltage source alone delivers 1 A and furnishes 1 W. We might then mistakenly calculate the total power delivered to the resistor as 2 W. This is incorrect.

Each source provides 1 A, making the total current in the resistor 2 A. The power delivered to the resistor is therefore 4 W.

5.3 Source Transformations

5.3.1 Practical Voltage Sources

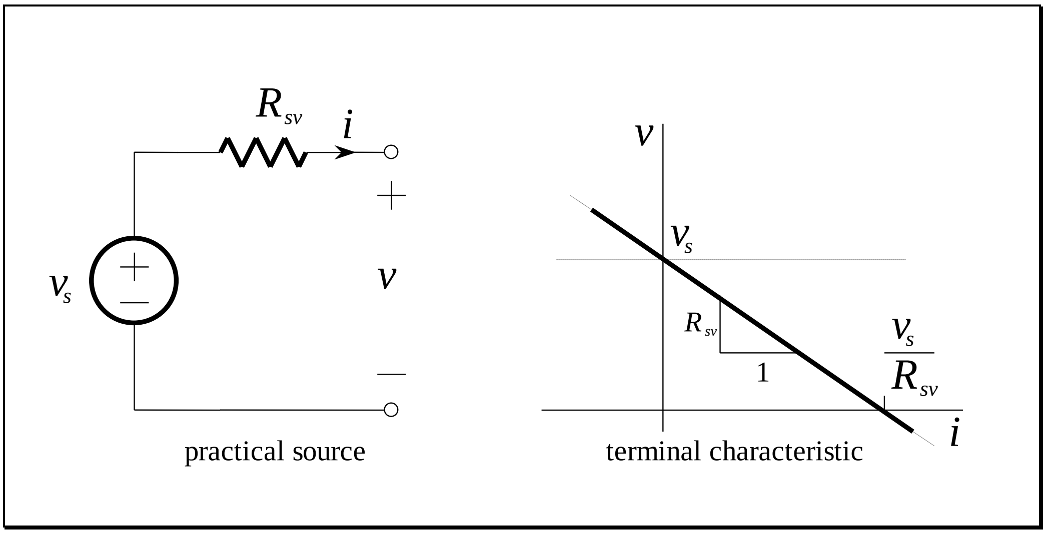
The ideal voltage source is defined as a device whose terminal voltage is independent of the current through it. Graphically, it's characteristic is:



An ideal voltage source, and its terminal characteristic

Figure 5.3

The ideal voltage source can provide any amount of current, and an unlimited amount of power. No such device exists practically. All practical voltage sources suffer from a voltage drop when they deliver current – the larger the current, the larger the voltage drop. Such behaviour can be *modelled* by the inclusion of a resistor in *series* with an ideal voltage source:



A practical voltage source, and its terminal characteristic

Figure 5.4

The terminal characteristic of a practical voltage source…

The terminal characteristic of the practical voltage source is given by KVL:

$$\boxed{v = v_s - R_{sv} i} \tag{5.2}$$

…shows the effect of the internal resistance

The resistance R_{sv} is known as the *internal resistance* or *output resistance*. This resistor (in most cases) is not a real physical resistor that is connected in series with a voltage source – it merely serves to account for a terminal voltage which decreases as the load current increases.

The applicability of this model to a practical source depends on the device and the operating conditions. For example, a DC power supply such as found in a laboratory will maintain a linear relationship in its terminal characteristic over a larger range of currents than a chemical battery.

When we attach a load to a practical voltage source:

A load attached to a practical voltage source will always exhibit less voltage and current than the ideal case

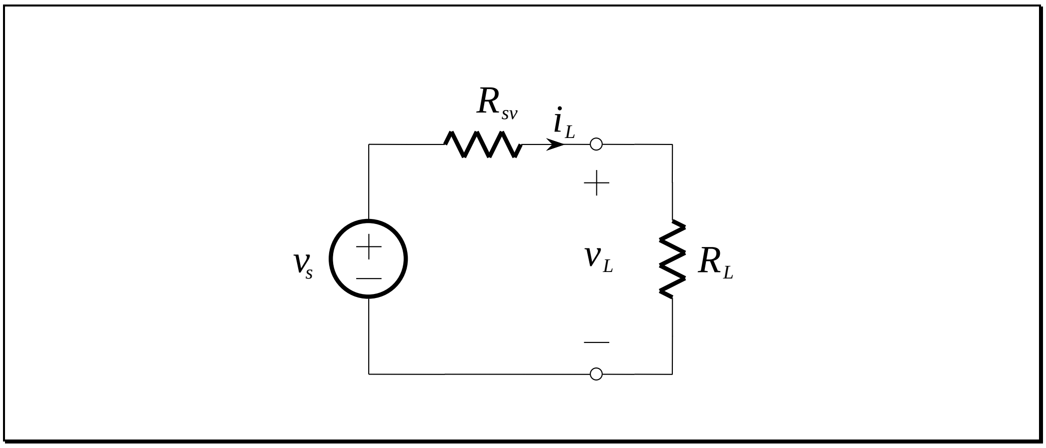


Figure 5.5

we get a load voltage which is always less than the open-circuit voltage, and given by the voltage divider rule:

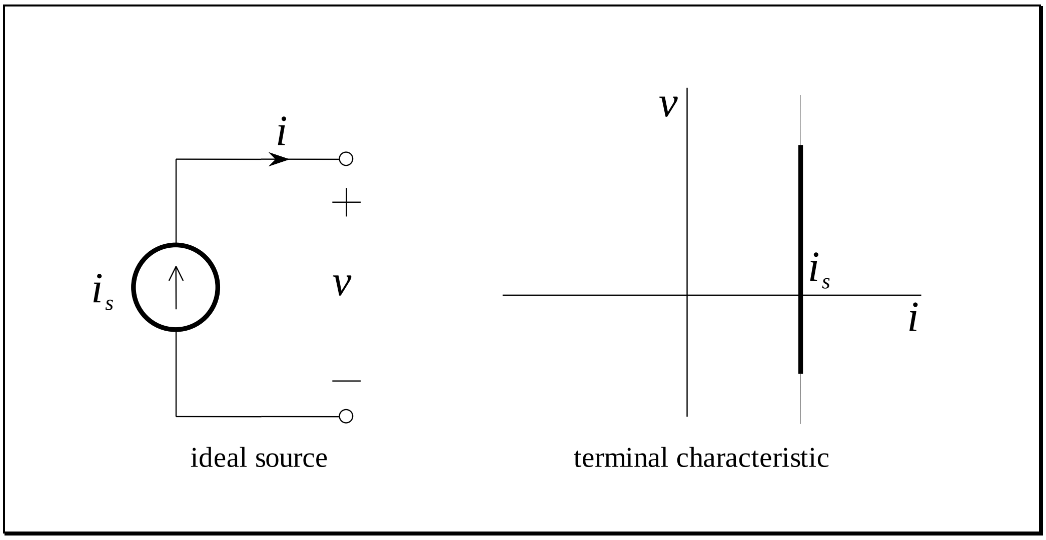
$$v_L = \frac{R_L}{R_{sv} + R_L} v_s < v_s \tag{5.3}$$

The load current will also be less than we expect from an ideal source:

$$i_L = \frac{v_s}{R_{sv} + R_L} < \frac{v_s}{R_L} \tag{5.4}$$

5.3.2 Practical Current Sources

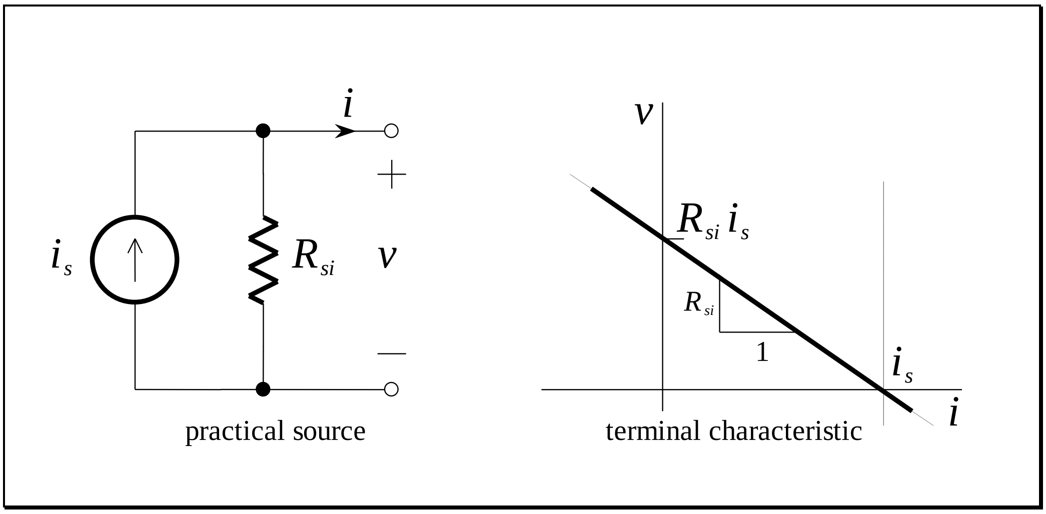
The ideal current source is defined as a device whose current is independent of the voltage across it. Graphically, it's characteristic is:



An ideal current source, and its terminal characteristic

Figure 5.6

The ideal current source can support any terminal voltage regardless of the load resistance to which it is connected, and an unlimited amount of power. An ideal current source is nonexistent in the real world. For example, transistor circuits and op-amp circuits can deliver a constant current to a wide range of load resistances, but the load resistance can always be made sufficiently large so that the current through it becomes very small. Such behaviour can be *modelled* by the inclusion of a resistor in *parallel* with an ideal current source:



A practical current source, and its terminal characteristic

Figure 5.7

5.12

The terminal characteristic of a practical current source

The terminal characteristic of the practical current source is given by KCL:

$$i = i_s - \frac{v}{R_{si}}$$

$$\text{or} \quad v = R_{si} i_s - R_{si} i \tag{5.5}$$

A load attached to a practical current source will always exhibit less voltage and current than the ideal case

When we attach a load to a practical current source:

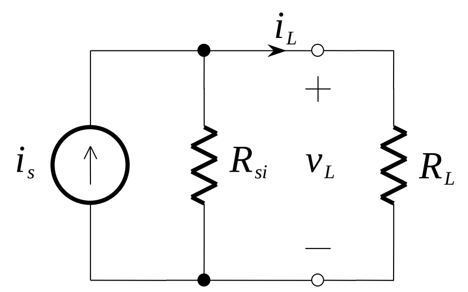


Figure 5.8

we get a load current which is always less than the short-circuit current, and given by the current divider rule:

$$i_L = \frac{R_{si}}{R_{si} + R_L} i_s < i_s \tag{5.6}$$

The load voltage will also be less than we expect from an ideal source:

$$v_L = \frac{R_{si} R_L i_s}{R_{si} + R_L} < R_L i_s \tag{5.7}$$

5.3.3 Practical Source Equivalence

We define two sources as being *equivalent* if each produces identical current and identical voltage for *any* load which is placed across it terminals. With reference to the practical voltage and current source terminal characteristics:

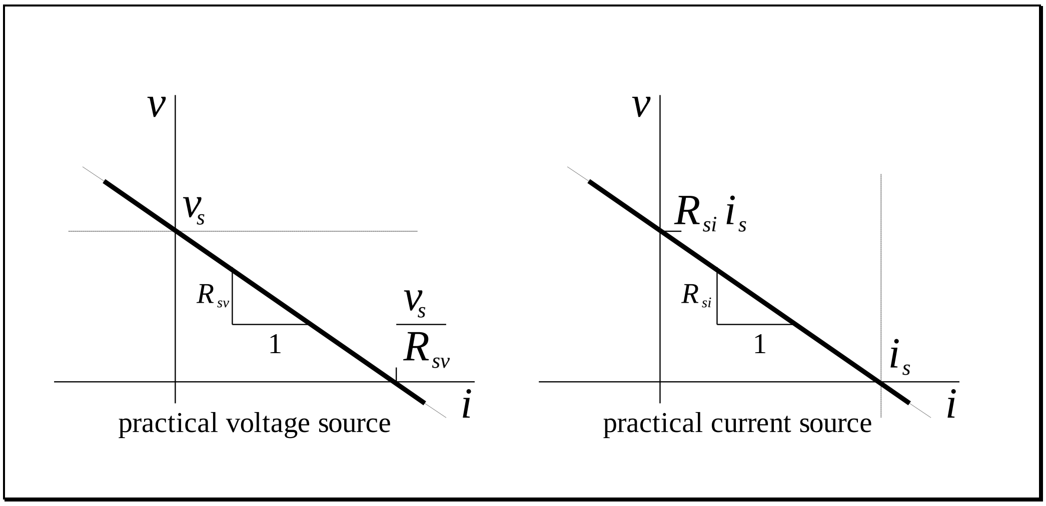


Figure 5.9

we can easily establish the conditions for equivalence. We must have:

$$\boxed{R_{sv} = R_{si} = R_s} \tag{5.8}$$

If practical sources are equivalent then they have the same internal resistance

so that the slopes of the two terminal characteristics are equal. We now let R_s represent the internal resistance of either practical source. To achieve the same voltage and current axes intercepts, we must have, respectively:

$$v_s = R_{si} i_s \quad \text{and} \quad \frac{v_s}{R_{sv}} = i_s \tag{5.9}$$

But since $R_{sv} = R_{si} = R_s$, these two relations turn into just one requirement:

$$\boxed{v_s = R_s i_s} \tag{5.10}$$

The relationship between a practical voltage source and a practical current source

The equivalence of practical sources

We can now transform between practical voltage and current sources:

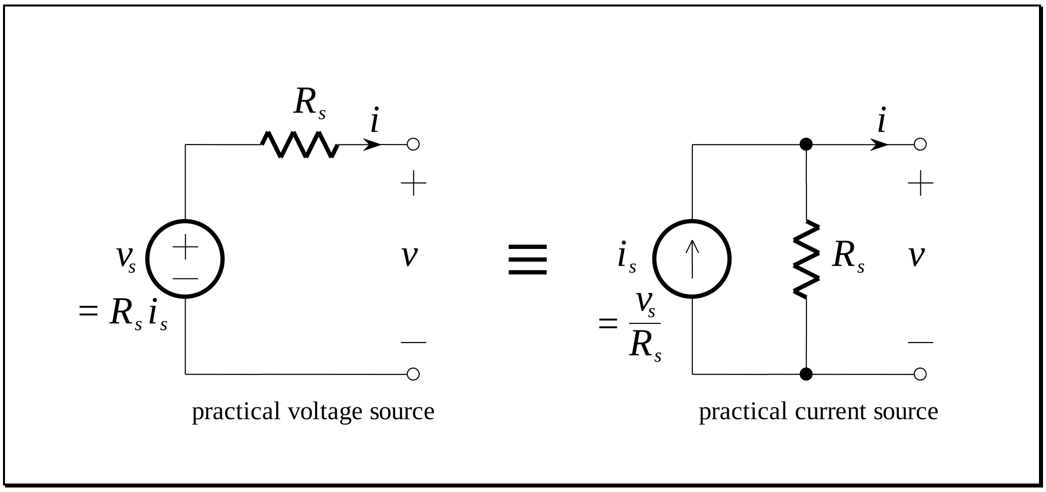
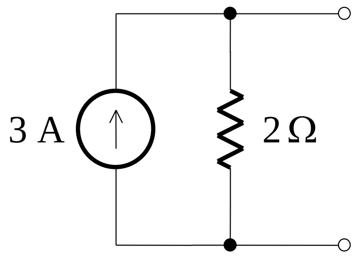


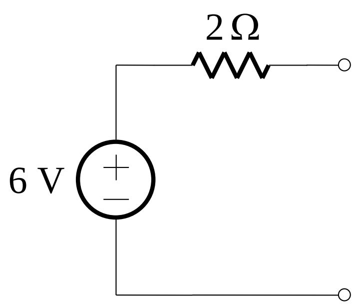
Figure 5.10

EXAMPLE 5.4 Equivalent Practical Sources

Consider the practical current source shown below:



Since its internal resistance is $2\,\Omega$, the internal resistance of the equivalent practical voltage source is also $2\,\Omega$. The voltage of the ideal voltage source contained within the practical voltage source is $v_s = R_s i_s = 2 \times 3 = 6\text{ V}$. The equivalent practical voltage source is shown below:



5.3.4 Maximum Power Transfer Theorem

Consider a practical DC voltage source:

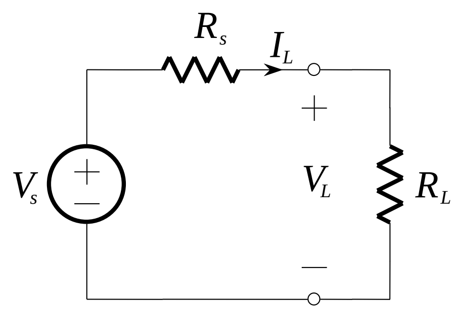


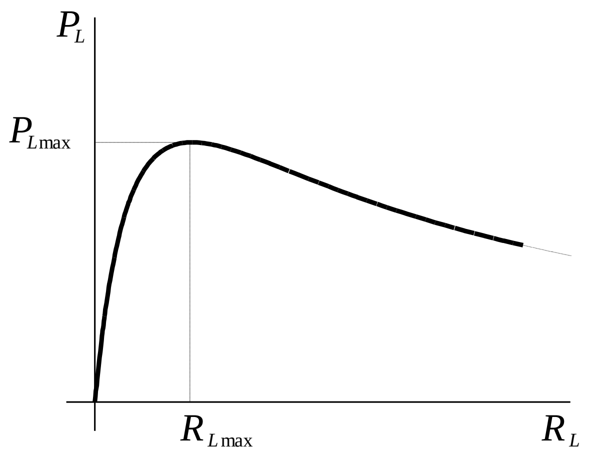
Figure 5.11

The power delivered to the load R_L is:

$$P_L = R_L I_L^2 = \frac{R_L V_s^2}{\left(R_s + R_L\right)^2} \tag{5.11}$$

Assume that V_S and R_S are known and fixed, and that R_L is allowed to vary.

A graph of the load power P_L versus load resistance R_L is shown below:



A graph of the power delivered to a load versus the load resistance shows clearly that a peak occurs at a certain resistance

Figure 5.12

To find the value of R_L that absorbs maximum power from the practical source, we differentiate with respect to R_L (using the quotient rule):

$$\frac{\partial P_L}{\partial R_L} = \frac{\left(R_s + R_L\right)^2 V_s^2 - V_s^2 R_L (2)\left(R_s + R_L\right)}{\left(R_s + R_L\right)^4} \tag{5.12}$$

and equate the derivative to zero to obtain the relative maximum:

$$\left(R_s + R_L\right)^2 - 2R_L\left(R_s + R_L\right) = 0 \tag{5.13}$$

or:

The load resistance which maximizes power delivered from a practical source

$$\boxed{R_L = R_s} \tag{5.14}$$

Since the values $R_L = 0$ and $R_L = \infty$ both give a minimum ($P_L = 0$), then this value is the absolute maximum (and not just a relative maximum).

Since we have already proved the equivalence between practical voltage and current sources, we have proved the following *maximum power transfer* theorem:

The maximum power transfer theorem…

> An independent voltage source in series with a resistance R_s, or an independent current source in parallel with a resistance R_s, delivers a maximum power to that load resistance R_L when $R_L = R_s$. (5.15)

…only applies to a choice of load resistor

We can only apply the maximum power transfer theorem when we have control over the load resistance, i.e. if we know the source resistance, then we can choose $R_L = R_s$ to maximize power transfer. On the other hand, if we are given a load resistance and we are free to design or choose a source resistance, we do **not** choose $R_s = R_L$ to maximize power transfer – by examining Eq. (5.11), we see that for a voltage source we should choose $R_s = 0$ (and for a current source we should choose $R_s = \infty$).

If we choose $R_L = R_s$ to obtain maximum power transfer to a load, then by Eq. (5.11) that maximum power is:

$$P_{L\,\max} = \frac{V_s^2}{4R_L} = \frac{V_s^2}{4R_s} \tag{5.16}$$

The maximum power delivered from a practical source

There is a distinct difference between *drawing* maximum power from a source and *delivering* maximum power to a load. If the load is sized such that $R_L = R_s$, it will receive maximum power from that source. However, considering just the practical source itself, we draw maximum possible power from the ideal voltage source by drawing the maximum possible current – which is achieved by shorting the source's terminals. However, in this extreme case, we *deliver* zero power to the "load" (a 0 Ω resistor).

Power matching is used in three situations:

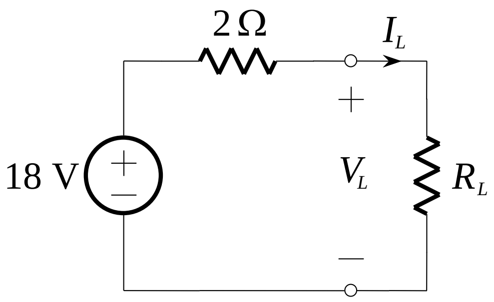
- where the signal levels are very small, so any power lost gives a worse signal to noise ratio. e.g. in antenna to receiver connections in television, radio and radar.

- high frequency electronics

- where the signal levels are very large, where the maximum efficiency is desirable on economic grounds. e.g. a broadcast antenna, audio amplifier.

EXAMPLE 5.5 Power Transfer

Consider the circuit shown below:



We want to determine the values of the load resistor that draw *half* the maximum power deliverable by the practical source. The maximum power deliverable by the source is:

$$P_{L\max} = \frac{V_s^2}{4R_s} = \frac{18^2}{4\times 2} = 40.5\text{ W}$$

Half the maximum power deliverable is therefore 20.25 W. The power dissipated by the load resistor is:

$$P_L = R_L I_L^2 = R_L\left(\frac{V_s}{R_s + R_L}\right)^2$$

Substituting values gives:

$$\begin{aligned} 20.25 &= R_L\left(\frac{18}{2+R_L}\right)^2 = \frac{324R_L}{(2+R_L)^2} \\ (2+R_L)^2 &= 16R_L \\ R_L^2 - 12R_L + 4 &= 0 \end{aligned}$$

Solving this quadratic gives:

$$\begin{aligned} R_L &= \frac{12 \pm \sqrt{12^2 - 4\times 4}}{2} \\ &= 6 \pm \sqrt{32} \\ &= 11.66 \text{ or } 0.3431\ \Omega \end{aligned}$$

5.4 Summary

- A linear circuit is one that contains linear elements, independent sources, and linear dependent sources. For a linear circuit, it is possible to show that "the response is proportional to the source".

- The superposition theorem states that, in evaluating the "response" in a linear circuit due to several sources, we are free to treat each independent source separately, collectively, or in any number of parts, and then superpose the response caused by each part.

- A practical voltage source consists of an ideal voltage source v_s in series with a resistance R_{sv}. A practical current source consists of an ideal current source i_s in parallel with a resistance R_{si}. The practical sources can be made equivalent by setting $R_{sv} = R_{si} = R_s$ and $v_s = R_s i_s$.

- The maximum power transfer theorem states that if we know the source resistance R_s of a practical source, then to maximize power transfer to a load R_L, we set $R_L = R_s$.

5.5 References

Hayt, W. & Kemmerly, J.: *Engineering Circuit Analysis*, 3rd *Ed.*, McGraw-Hill, 1984.

Exercises

1.

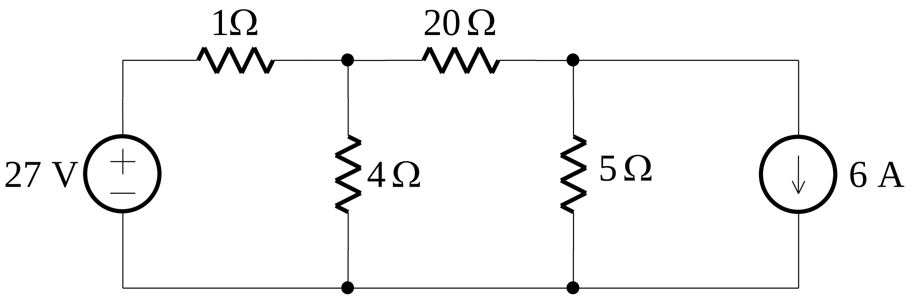
Find the power dissipated in the $20\,\Omega$ resistor of the circuit shown below by each of the following methods:

(a) nodal equations

(b) mesh equations

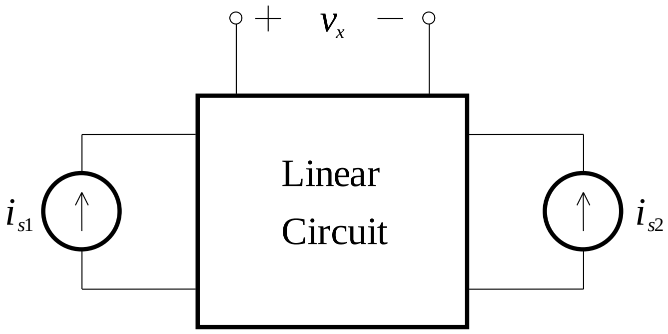
(c) source transformations to eliminate all current sources, then a method of your choice

(d) source transformations to eliminate all voltage sources, followed by any method you wish.



2.

Consider the linear circuit shown below.



(a) The circuit contains only resistors. If $i_{s1} = 8\ \text{A}$ and $i_{s2} = 12\ \text{A}$, v_x is found to be 80 V. However, if $i_{s1} = -8\ \text{A}$ and $i_{s2} = 4\ \text{A}$, then $v_x = 0\ \text{V}$. Find v_x when $i_{s1} = i_{s2} = 20\ \text{A}$.

(b) The circuit now contains a source such that $v_x = -40\ \text{V}$ when $i_{s1} = i_{s2} = 0\ \text{A}$. All data in part (a) are still correct. Find v_x when $i_{s1} = i_{s2} = 20\ \text{A}$.

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