11 Potential Difference

Contents

11.2

Introduction

Potential difference is a fundamental concept to the electrical engineer. From a field perspective, a knowledge of the "potential" V at every point in space is enough to deduce the electric field **E** at every point in space. From a circuit perspective, it is necessary to understand potential difference, electric fields, and flux densities in order to understand and characterise the behaviour of a new circuit element called the *capacitor*.

We will formulate the concept of potential difference from the basic definition of work done per unit charge. This will lead to new insights into the **E** field, such as discovering that it is a *conservative* field, and potential difference is independent of the path taken in order to arrive at a different potential.

The concept of absolute potential will be introduced, and then the concept of an equipotential surface. This gives rise to field plots that include equipotential "contour lines (or surfaces)" that can aid in understanding the pattern of electric flux lines.

Next we introduce the key concept of *potential gradient*. For simple 2D geometries, this gives rise to a picture of potential as a surface, and provides a new perspective on how charges move in an **E** field.

Lastly, we will consider how we can produce an electrostatic field plot, complete with flux lines and equipotentials, from just a few simple rules.

11.1 Potential Difference

Electric potential difference between two points *A* and *B* involves a calculation of the amount of work done in moving a test charge between the points *A* and *B*. We have just seen that an electric charge situated in an electric field will experience a force which is electrical in origin (Coulomb's Law). If the charge is allowed to move under the influence of this force, the field may be seen to be doing work. Likewise, if we as an observer cause the charge to move against the electric force we must supply a mechanical force which opposes and overcomes the electric Coulomb force. If our mechanical force is just equal to the electric force then the charge will move without acceleration, that is with infinitesimally small velocity, and none of the work which we have supplied in order to move the charge will appear as kinetic energy. Also, if the motion occurs without friction, the mechanical work done by us in moving the charge with infinitesimally small velocity may be recovered from the system by allowing the field to do work on the charge. In other words we have increased the potential energy of the charge. This increase (or decrease) in potential energy is measured in terms of a quantity called *electric potential difference*. Formally the definition of electric potential difference is as follows:

Electric potential difference between two points *A* and *B*
is defined as the mechanical work done per unit charge (11.1)
in moving a test charge from point *A* to point *B*.

From our foregoing discussion we might also define the electric potential difference in terms of the electrical work done per unit charge performed by the field in moving the test charge from point *B* to point *A*.

Assume that we have two points *A* and *B* located in an electric field as shown below:

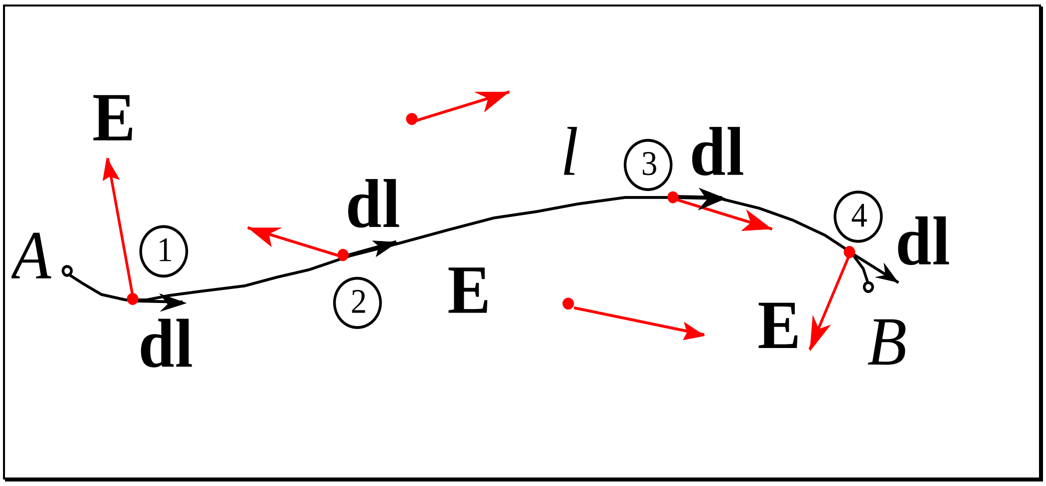


Figure 11.1 – Illustrating a path of integration between two points

In this figure, the electric field **E** (shown as red vectors) is a function of position. It changes both its magnitude and direction from place to place. An arbitrary path from *A* to *B* is shown – it need not be a straight line. Only the value of **E** along the path is important, since that is what determines the electric force on the test charge. We need to apply a mechanical force to either push or pull the charge along its journey from *A* to *B*. Sometimes the electric field will be aiding us, sometimes it will be resisting us.

Four points along the path are shown. At point 1, the electric field is almost at right angles to the direction of our path, so the mechanical work required here will be small. At point 2 the electric field will be trying to push our test charge back as we move along the path – we will have to put in mechanical work to get past this point. At point 3 the electric field is almost in alignment with our path and will be aiding us. At point 4 the electric field is at a perfect right angle to our path, and we don't need to provide any mechanical force to move.

Note that we have broken the path *l* up into infinitesimally small directed pieces (vectors) represented by **dl**. At all points along the path, **dl** is tangential (points in the direction of the tangent to the curve).

Remember that to move a charge infinitesimally slowly in an electric field, we apply a mechanical force that exactly counteracts the Coulomb force. A larger force than the Coulomb force will accelerate the charge. To overcome the electric force, we just need to supply a mechanical force given by:

$$\mathbf{F}_M = -\mathbf{F}_E \tag{11.2}$$

The amount of mechanical work we have to do to move a test charge along the infinitesimally small **dl** is then given by:

Work done depends on the electric field

$$dW = \mathbf{F}_M \cdot \mathbf{dl} = -\mathbf{F}_E \cdot \mathbf{dl} \tag{11.3}$$

Noting that Coulomb's Law can be written:

$$\mathbf{F}_E = q\mathbf{E} \tag{11.4}$$

we get:

$$dW = -q\mathbf{E} \cdot \mathbf{dl} \tag{11.5}$$

The above equation gives the work done when moving an infinitesimal displacement **dl**. To find the work done in moving the charge along the entire path from *A* to *B*, we perform what is known as a *line integral*:

$$W_{BA} = -q\int_A^B \mathbf{E} \cdot \mathbf{dl} \quad \text{J} \tag{11.6}$$

Work done for an arbitrary path in an electric field

Potential difference is the mechanical work done per unit charge, so the potential difference between two points *A* and *B* in an electric field is:

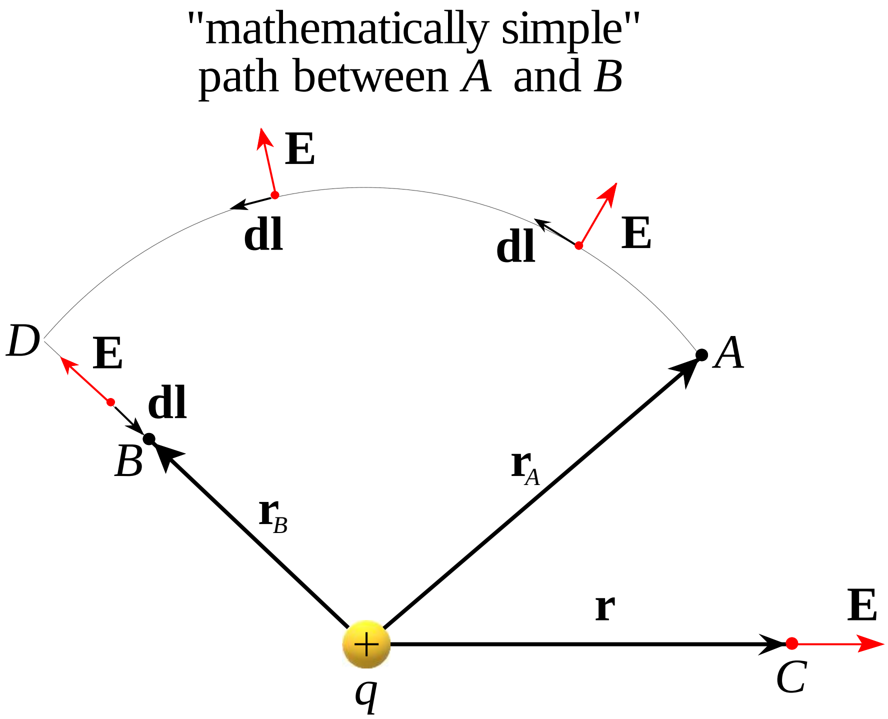
$$V_{BA} = -\int_A^B \mathbf{E} \cdot \mathbf{dl} \quad \text{V} \tag{11.7}$$

Potential difference defined

It is interesting that this integral is *independent* of path *l* taken from *A* to *B*. (A formal proof of this statement is beyond the scope of this topic. However, a reference to the mechanical inclined plane may make you happier to accept it unproven). Therefore, in determining V_{BA} it is usual to choose the mathematically simplest path available for *l*.

EXAMPLE 11.1 Potential Difference in a Field Due to a Point Charge

As an example of the calculation of the electrical potential difference between two points in an electric field, using an electric field that we are able to calculate, assume that an isolated point charge of $+q$ coulombs is located in free space, as shown in the figure below:



We know that the electric field at any point C distance r from the point charge is given by:

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$

Interpreting this equation, we see that **E** is directed radially away from the point charge q throughout the whole of space (3-dimensional). Assume for simplicity that all points of interest in our example are in the same plane as shown in the figure.

To determine the potential difference between the points B and A in the field, we must firstly choose a path l along which to integrate so that V_{BA} may be determined reasonably simply. If we choose a path directly $A \rightarrow B$ it is possible to deduce the integral, but it involves an interpretation of the scalar product (dot product) of two vectors which will include the cosine of the angle between them. A simpler approach would be to realise that the integral is *path independent* and that we may choose any path for the integration that suits us. One such path would be the path *ADB* in the figure.

Along the portion of the path *A*→*D*, the angle between **E** and **dl** will be 90°. Therefore:

$$\mathbf{E}\cdot\mathbf{dl}\big|_{A\to D} = Edl\cos 90° = 0$$

We conclude that it takes no work to move in a circle around a point charge (it takes no work to walk around the side of a hill at a constant height).

Along the portion of the path *D*→*B*, the angle between **E** and **dl** will be 180°. Therefore:

$$\mathbf{E}\cdot\mathbf{dl}\big|_{D\to B} = Edl\cos 180° = -Edl$$

The potential difference between *A* and *B* is then given by:

$$\begin{aligned}
V_{BA} &= -\int_A^B \mathbf{E}\cdot\mathbf{dl} \\
&= -\left(\int_A^D \mathbf{E}\cdot\mathbf{dl} + \int_D^B \mathbf{E}\cdot\mathbf{dl}\right) \\
&= -\left(0 + \int_D^B \mathbf{E}\cdot\mathbf{dl}\right) \\
&= -\int_D^B \mathbf{E}\cdot\mathbf{dl} \\
&= \int_D^B Edl
\end{aligned}$$

Now, along the path *D*→*B*, "a little bit of path length, *dl*" has the same magnitude but the opposite sign as "a little bit of radius, *dr*". Thus we have:

$$dl = -dr$$

This is because dr is defined to point radially outward (similar to dx being a small increment to the right on an x-axis), but our path dl points radially inward.

The potential difference between A and B is then given by:

$$V_{BA} = \int_{D}^{B} E dl = -\int_{r_A}^{r_B} E dr$$

Therefore:

$$\begin{aligned} V_{BA} &= -\int_{r_A}^{r_B} \frac{q}{4\pi\varepsilon_0 r^2} dr \\ &= -\frac{q}{4\pi\varepsilon_0} \int_{r_A}^{r_B} \frac{dr}{r^2} \\ &= \frac{q}{4\pi\varepsilon_0} \left[\frac{1}{r} \right]_{r_A}^{r_B} \\ &= \frac{q}{4\pi\varepsilon_0} \left[\frac{1}{r_B} - \frac{1}{r_A} \right] \end{aligned}$$

This tells us that the potential difference is positive if $r_B < r_A$, i.e. the potential closer to the charge (at radius r_B) is higher than the potential further away from the charge (at r_A).

11.2 Conservative Fields

It is apparent from the definition of potential difference that if we start at a point A in an **E** field, and follow an arbitrary path through the **E** field back to the starting point A, then there is no potential difference. In other words, it takes no *net* energy to return to a starting position. We may be required to supply energy at some point in our journey (to increase our potential), but when we return to our starting point we will recover the energy. The same is true for the gravitational field, if we neglect friction. If you walk up a hill and return to your starting position, there is no *net* energy expended. For such fields, we say the field is conservative. Expressing this mathematically for the **E** field:

$$\oint_l \mathbf{E} \cdot d\mathbf{l} = 0 \tag{11.8}$$

The electrostatic **E** field is conservative

The circle on the integral means "a closed path", and the subscript l on the integral denotes that the path is called l.

As you will discover in further study, conservative vector fields can be expressed entirely by a *scalar potential field*. This makes solving problems in electrostatics much easier, since we can transform *all* problems into finding a *scalar* field (such as electric potential) instead of trying to find a *vector* field (such as **E**).

11.3 Absolute Potential

If, in the preceding example, the radius vector $\mathbf{r}_A$ is assumed to stretch to infinity such that $\mathbf{r}_A \to \infty$ then the potential difference V_{BA} is associated with the amount of energy per unit charge involved in bringing a test charge from infinity to the point *B*. It represents the *absolute electric potential* at *B* in the field and it is written simply as:

Absolute potential for a point charge

$$V_B = \frac{q}{4\pi\varepsilon_0 r_B} \tag{11.9}$$

If r_B approaches infinity, the absolute potential V_B will approach zero. Hence points at an infinite distance from an isolated electrostatic charge will be located in a region of zero electrostatic potential (by definition).

The potential difference between two points in a field is given by the difference between their absolute potentials:

Potential difference for a point charge

$$\begin{aligned} V_{BA} &= V_B - V_A \\ &= \frac{q}{4\pi\varepsilon_0 r_B} - \frac{q}{4\pi\varepsilon_0 r_A} \\ &= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right] \end{aligned} \tag{11.10}$$

11.4 Equipotentials

An equipotential surface is defined as being a surface which joins places of equal electrostatic potential.

By inspection of Eq. (11.9), every point at a distance r_B from the isolated point charge $+q$ in the example will be at the same potential. That is, all points distance r_B from the charge will lie on an equipotential surface which in this case will be spherical.

Equipotential lines drawn on a field map resemble elevation lines on a topographic map. Equipotential lines, by definition, must always be perpendicular to field lines. An example of equipotentials drawn on a field map is shown below:

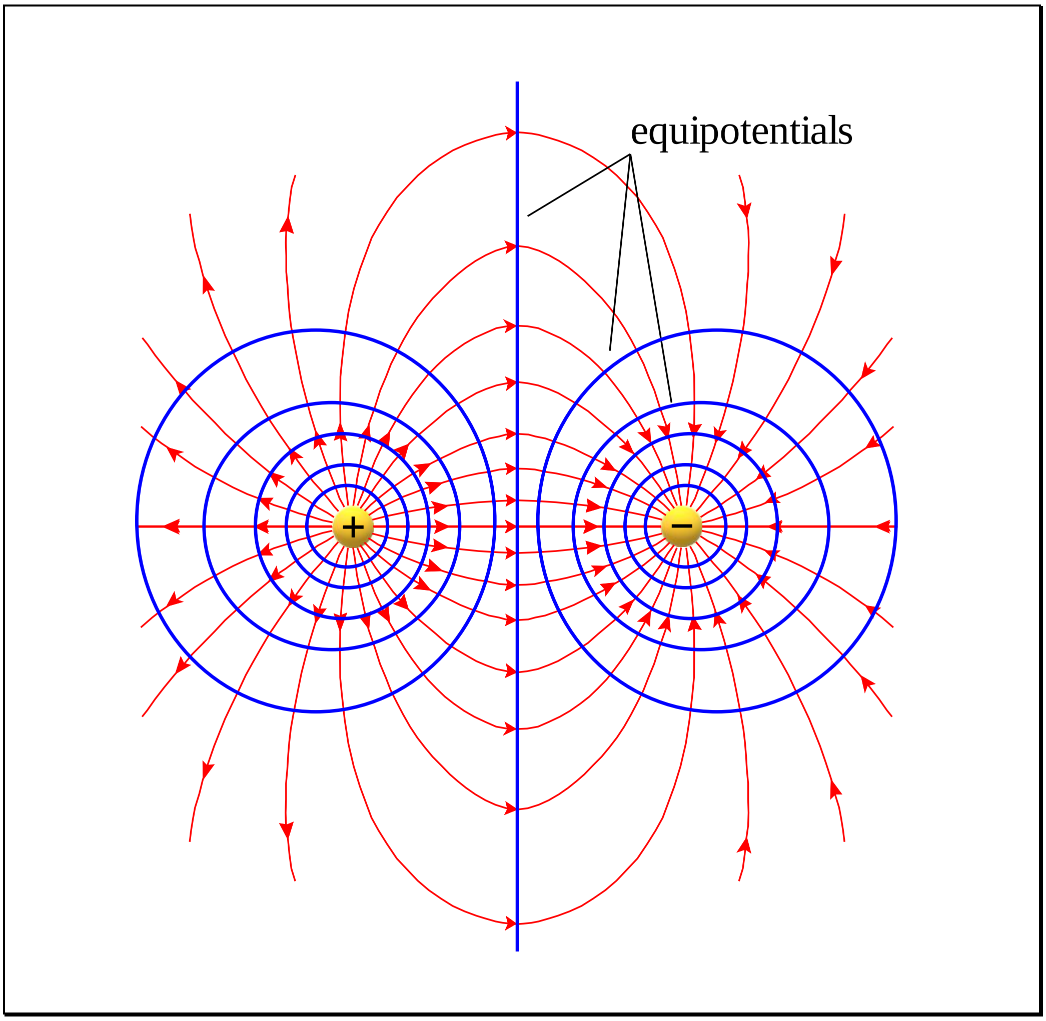


Figure 11.2 – Equipotentials and field lines of an electric dipole

11.5 Potential Gradient

Consider a uniform **E** field in a region between two infinite conducting plates (equipotentials), separated by a distance *l*, with a voltage V_{BA} across them:

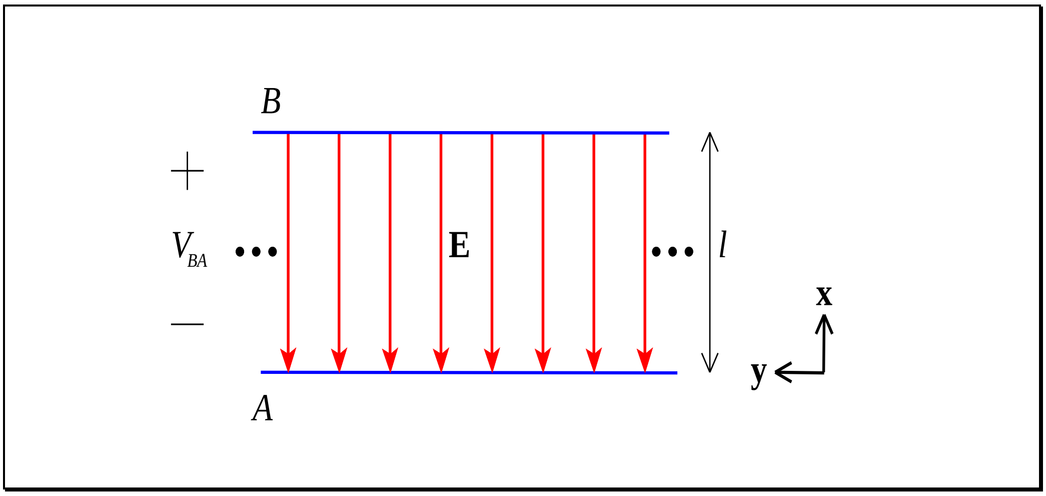


Figure 11.3 – A uniform E field

Since:

$$V_{BA} = -\int_A^B \mathbf{E} \cdot \mathbf{dl} \tag{11.11}$$

then in a uniform field (**E** is constant) in which the path from *A* to *B* follows the opposite direction to one of the straight electric field lines (**dl** is constant), the integral evaluates to a simple multiplication:

$$V_{BA} = El \tag{11.12}$$

Note that in evaluating the dot product, we used $\mathbf{d\hat{l}} = -\mathbf{\hat{E}}$ (we move from *A* to *B* in the opposite direction to **E**) – hence the dot product will be negative, and this will cancel the negative sign out the front of the integral. For a uniform field, the magnitude of the electric field is then just:

$$E = \frac{V}{l} \quad \text{Vm}^{-1} \tag{11.13}$$

Therefore, the electric field has units of volts/metre, and this is the unit used most commonly in practice rather than newtons/coulomb.

Now if we are to generalise Eq. (11.13) to **E** fields that are **not** uniform (i.e. vary from point to point), then we convert the formula to a differential form:

$$E = \frac{dV}{dl} \tag{11.14}$$

This equation is for the magnitude of **E**, but Eq. (11.11) is an equation involving vectors. These are directed quantities and it might therefore be expected that when reorganising Eq. (11.11) into a differential form, we will have to provide some direction information.

Let us denote the direction at which the rate of change of voltage is a maximum as $\hat{\mathbf{n}}$. Then at any point in an electric field, the direction of $\hat{\mathbf{n}}$ changes and points in the direction at which the voltage is rising fastest with increasing distance. The **E** field must then point in the opposite direction to $\hat{\mathbf{n}}$, since **E** points from a higher potential to a lower potential (a test charge in an **E** field will be accelerated to a lower potential).

Therefore, the differential form of $V_{BA} = -\int_A^B \mathbf{E} \cdot \mathbf{dl}$ is:

$$\mathbf{E} = -\frac{dV}{dl}\hat{\mathbf{n}} \tag{11.15}$$

where the derivative $\frac{dV}{dl}$ must be taken in the direction of $\hat{\mathbf{n}}$, i.e. in the direction which gives the maximum rate of change.

11.14

The figure below represents a map of the electrostatic potential of the field considered in Figure 11.3, where V is graphed as a 2-dimensional function of position in the x-y plane:

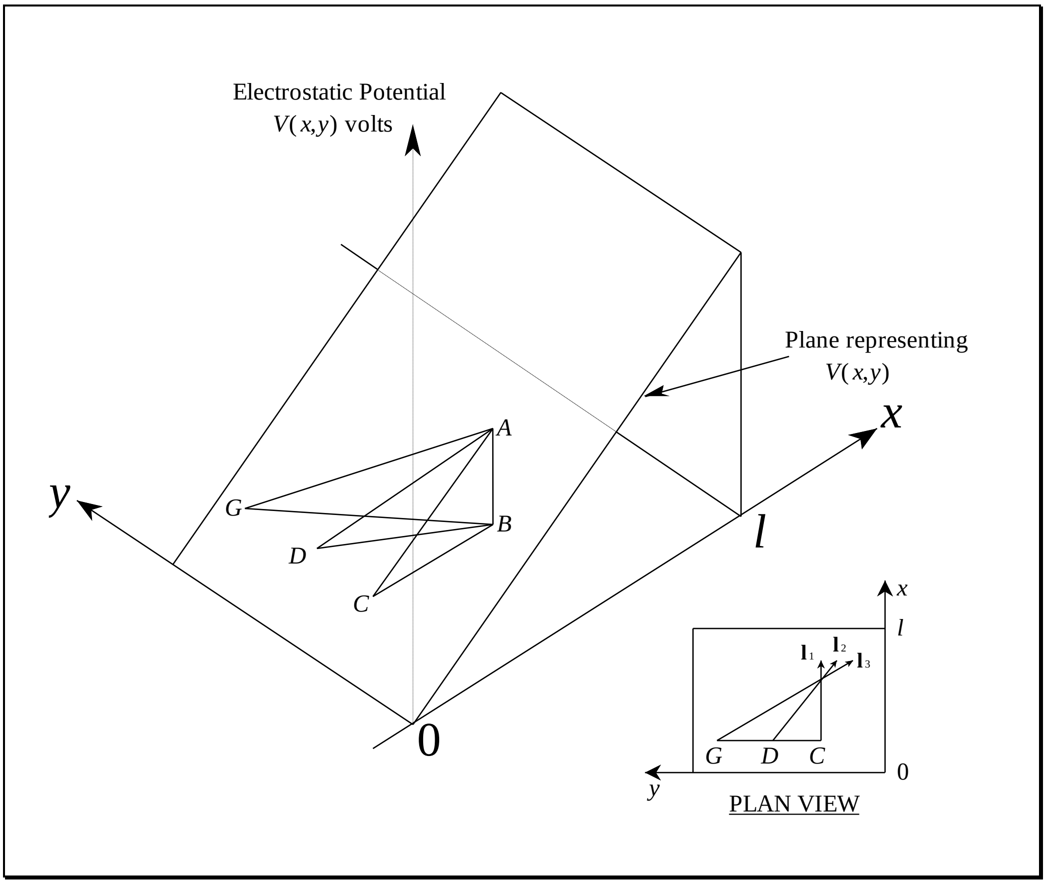


Figure 11.4 – A uniform E field

As shown, it appears that V increases uniformly in the x-direction. We can find the rate of increase of potential with position in a given direction by taking the slope of the plane $\frac{dV}{dl}$ in the direction under consideration. The figure shows constructions for determining the rate of change of potential with position in three different directions. You will notice that the maximum rate of change of potential with position will be obtained when we choose a direction $\mathbf{l}_1$ which moves us directly up the potential 'hill'. The other two rates of change shown in directions $\mathbf{l}_2$ and $\mathbf{l}_3$ give us rates of change which are less than the maximum.

By definition, the *gradient* of the potential is defined by the rate of *increase* of potential with position obtained when we move in the direction of *maximum rate of increase* – that is, we move directly *up* the potential hill. You might like to compare this definition with that relating to the gradient of a roadway moving directly up a hill.

If a positive test charge is released from rest on the potential hill, it will move towards a position of lower potential – that is, it will move directly down the hill. The direction of the electric field represented by the potential distribution shown in Figure 11.4 will be directly *down* the potential hill. The direction of the electric field is therefore directly opposite the direction of the gradient of the field – hence the negative sign in Eq. (11.15). In the example shown in Figure 11.4:

$$E = \left.\frac{dV}{dl}\right|_{\max} = \frac{dV}{dx} \tag{11.16}$$

Since the direction of maximum rate of change of voltage with position is in the $\hat{\mathbf{x}}$ direction, this must be the direction of $\hat{\mathbf{n}}$. Therefore, in the example, we have:

$$\mathbf{E} = -\frac{dV}{dx}\hat{\mathbf{x}} \tag{11.17}$$

11.6 Plots of Potential Difference

If we restrict our considerations to a 2D *x*-*y* plane, we can plot potential difference as a surface above the plane where the height of the surface is equal to the potential difference. A plot of this type for an electric dipole looks like:

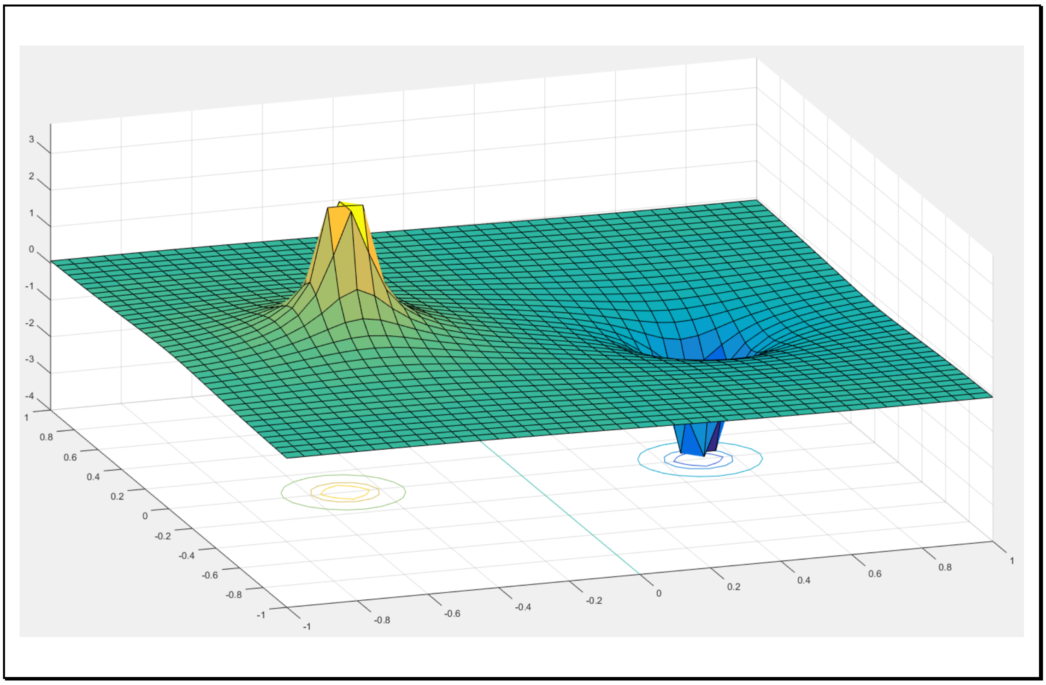


Figure 11.5 – A plot of potential difference for an electric dipole

In this plot, we have arbitrarily set 0 V midway between the two charges. The charge on the left is positive, and the potential hill rises up to infinity as we approach the point charge. The charge on the right is negative, and the potential valley falls to negative infinity as we approach the point charge.

Below the surface we have drawn a "contour plot", which shows equipotential lines. Compare the contour plot with that shown in Figure 11.2.

With this pictorial view of potential difference, we can now imagine how a test charge moves in an electric field – place your test charge anywhere on the surface, and then watch it "roll away" – it will follow the path which has the maximum downhill gradient, just like a ball rolling down a hill in a gravitational field. The **E** field in this pictorial view will be vectors at each point on the surface, with magnitude equal to the slope of the surface, and with a direction pointing "downhill".

11.7 Electrostatic Field Plotting

In order to plot electrostatic fields it is important to consider some of their properties. If electric charges of the same sign are placed onto a conductor they will initially move and re-arrange themselves on the surface of the conductor until each charge is situated such that the component of the electric field intensity which is tangential to the surface of the conductor in the vicinity of each charge has been reduced to zero. This is necessary if the charge is to be *static* (stationary).

Likewise the charges will reside on the surface of a conductor since it is the surface location which gives the charges maximum physical separation. Electrostatic fields will be non-existent within conductors for this reason. To prove this use Gauss' Law and take your surface of integration just under the surface of the conductor. Since there is no charge enclosed, ψ and therefore **D** must be zero and hence $\mathbf{E} = 0$:

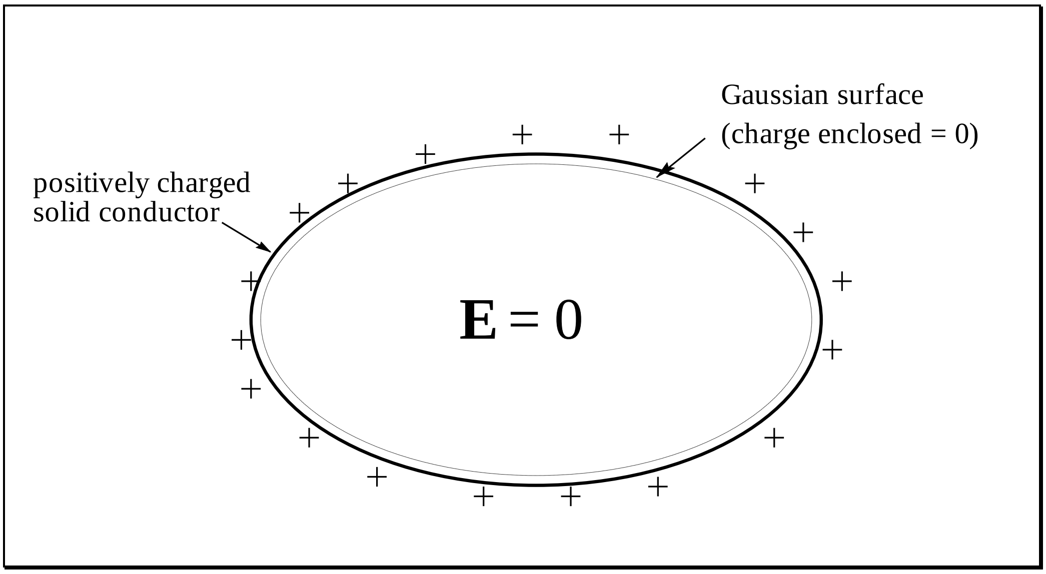


Figure 11.6 – Diagram showing the location of the Gaussian surface used to show that E = 0 within a solid conductor

It has been stated previously by way of definition that the electric field intensity is related to the negative of the gradient of the electrostatic potential. The gradient of the electrostatic potential is defined as being the rate of change of potential with distance where the rate at any point is taken in the direction where the rate is of maximum value. If you picture an arrangement of equipotential surfaces, the direction of maximum rate of change of potential with distance at a given point will be that direction perpendicular to the equipotential surface through the point. Hence the *direction of the electric field intensity vector* at the point will be *perpendicular to the equipotential surface* through the point.

The major points relating to field plotting are as follows:

1. Free electrostatic charges reside on the surface of conductors (the outside surface of ho1low conductors).

2. Electric flux lines leave the surface of *conductors perpendicularly* (no tangential component of **D** or **E**).

3. Electric flux lines and equipotential surfaces cross at right angles.

4. Lines of flux do not cross.

It is important to realise that lines of electric flux commence on positive charges and terminate on negative charges, even if it is necessary that some of these charges must reside at infinity.

Also, the assumption is made that matter is initially neutral, thus in order to create a negatively charged body, electrons must be removed from a second body and deposited on the first. The second body will therefore become positively charged in the process (excess protons). If the second body is then physically removed, we may consider it to be located at infinity.

11.8 Summary

- Potential difference is defined as the work done per unit charge in moving from A to B in an electric field:

$$V_{BA} = -\int_A^B \mathbf{E} \cdot \mathbf{dl} \quad \text{V}$$

- The **E** field is conservative, which means no work is done in moving a charge through an **E** field and returning to the starting point:

$$\oint_l \mathbf{E} \cdot \mathbf{dl} = 0$$

- The absolute potential at a point a distance r away from a point charge is given by:

$$V = \frac{q}{4\pi\varepsilon_0 r}$$

- An equipotential surface is defined as being a surface which joins places of equal electrostatic potential. In A 2D field plot, we have equipotential curves, which are similar to contour lines on a topographic map.

- The electric field **E** can be derived from the potential field as the negative of the *gradient*:

$$\mathbf{E} = -\frac{dV}{dl}\hat{\mathbf{n}}$$

  where the derivative $\frac{dV}{dl}$ must be taken in the direction of $\hat{\mathbf{n}}$ (which points in the direction which gives the maximum rate of change of potential).

- Electrostatic field plots can be sketched by following just a few simple rules.

11.9 References

Beard, G.: *Electrostatics and Magnetostatics*, NSWIT, 1985.

Bryce, P.: *Field Theory 2 Notes*, NSWIT, 1988.

Exercises

1.

Point charges of 2×10^{-19} C are situated at each of three corners of a square whose sides are 0.2 m. Calculate the electrostatic potential at the centre and vacant corner of the square.

2.

Calculate the voltage at a point (P), 30 mm from a positive charge of 8×10^{-6} C when this point (P) is also 60 mm away from a negative charge of -8×10^{-6} C. The two charges are separated by a distance of 90 mm.

3.

Derive expressions for the potential at a distance *r*, in a vacuum, from:

(a) the centre of a uniformly charged sphere, with radius *a*, and a total charge *q*, when $r \geq a$.

(b) an infinite line charge with uniform charge density λ Cm^{-1}.

(c) an infinite plane with uniform charge density σ Cm^{-2}.

In each case, sketch the field patterns with a constant potential difference between equipotentials.

4.

An insulated conductor is charged positively. How does the conductor potential vary when an earthed conducting plane is brought near it?

5.

How can an insulated charged conductor be shielded from the effects of neighbouring earthed or charged conductors?