12 Capacitance

Contents

Introduction

Capacitance is the ratio of the charge stored on one of two surfaces to the potential difference between them. It is a property of the geometry of the surfaces, and the insulating material between them. A field theory approach to describing capacitance gives rise to some simple formula which can then be used as an approximation to real-life devices that have been constructed to exploit the electric field between conductors – these devices are called *capacitors*.

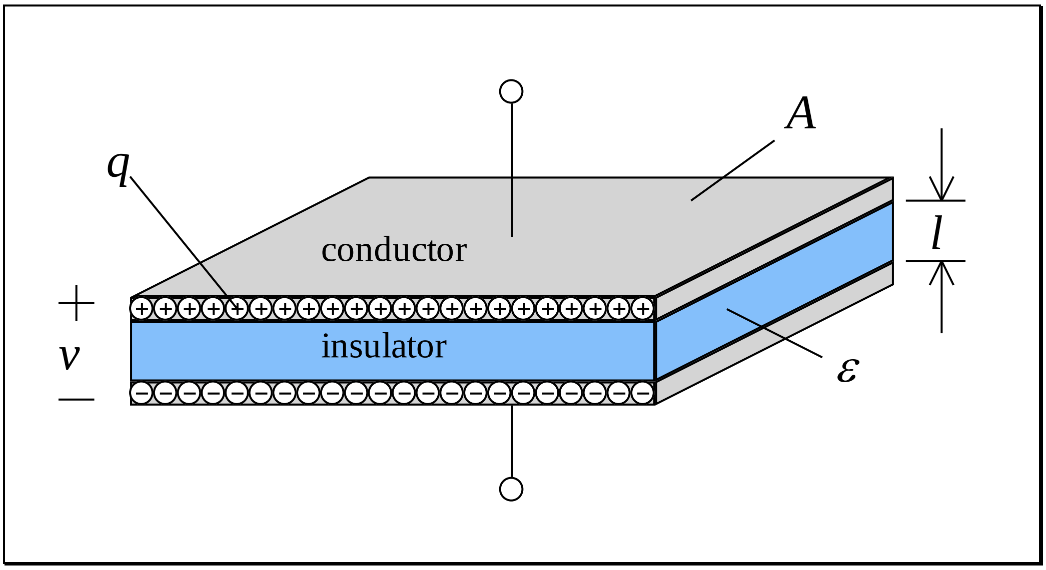
Physically, a capacitor consists of two conducting surfaces on which a charge may be stored, separated by a thin insulating layer which has a very large resistance. Energy is stored in the *electric field* that exists between the capacitor's two conducting surfaces. In addition, the insulating layer may be made of a high permittivity material (such as ceramic) which will dramatically increase the capacitance (compared to air).

When considering structures which exhibit capacitance, it will be necessary to resort to field theory in order to understand changes to the system such as increasing plate separation; changing the voltage; or introducing a new dielectric.

From a circuit perspective, the capacitor is a circuit element whose voltage-current relationship involves the rate of change of voltage. This relationship gives rise to unique properties of circuit behaviour, as will become evident with further study.

12.1 The Capacitor

The simplest capacitor is formed by two conductive plates separated by a dielectric layer:



A parallel plate capacitor

Figure 12.1

One of the plates carries a positive charge, q, whilst the other carries an equal but opposite charge, $-q$. Therefore, the capacitor *stores* charge. There is a potential difference, v, between the plates. Ideally, the amount of charge q deposited on the plates is proportional to the voltage v impressed across them. We define a constant[1] called the *capacitance*, C, of the structure by the linear relationship:

$$q = Cv \tag{12.1}$$

The definition of capacitance

The unit of capacitance is the *farad*, with symbol F.

There is an *electric field* between the capacitor plates which is established by the charges present on the plates. Energy is stored in the capacitor by virtue of this electric field.

[1] The constant only *models* the behaviour of the structure under certain operating conditions. The capacitance of a structure in the real world will vary with temperature, voltage, pressure, frequency, chemical aging, etc.

The ideal capacitance relationship is a *straight line through the origin*:

The capacitor is a linear circuit element

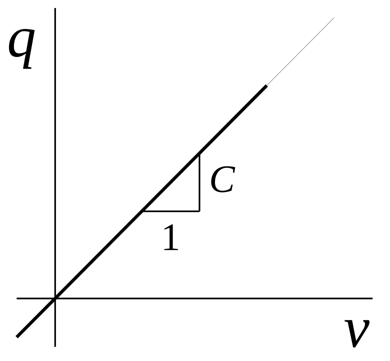


Figure 12.2

Even though capacitance is defined as $C = q/v$, it should be noted that C is a purely geometric property, and depends only on the conductor arrangements and the materials used in the construction.

12.2 Calculation of Capacitance

In order to calculate capacitance between two conducting bodies using our knowledge of electrostatic field theory, the ratio of the charge Q to the potential difference V between them must be calculable. To calculate V it is essential to know the shape of the field.

12.2.1 Capacitance Between Two Large, Closely-Spaced, Parallel Conducting Plates

Assume that the upper plate is charged positively with a charge of $+Q$ coulombs and that the lower plate is charged negatively with a charge of $-Q$ coulombs as in the figure below:

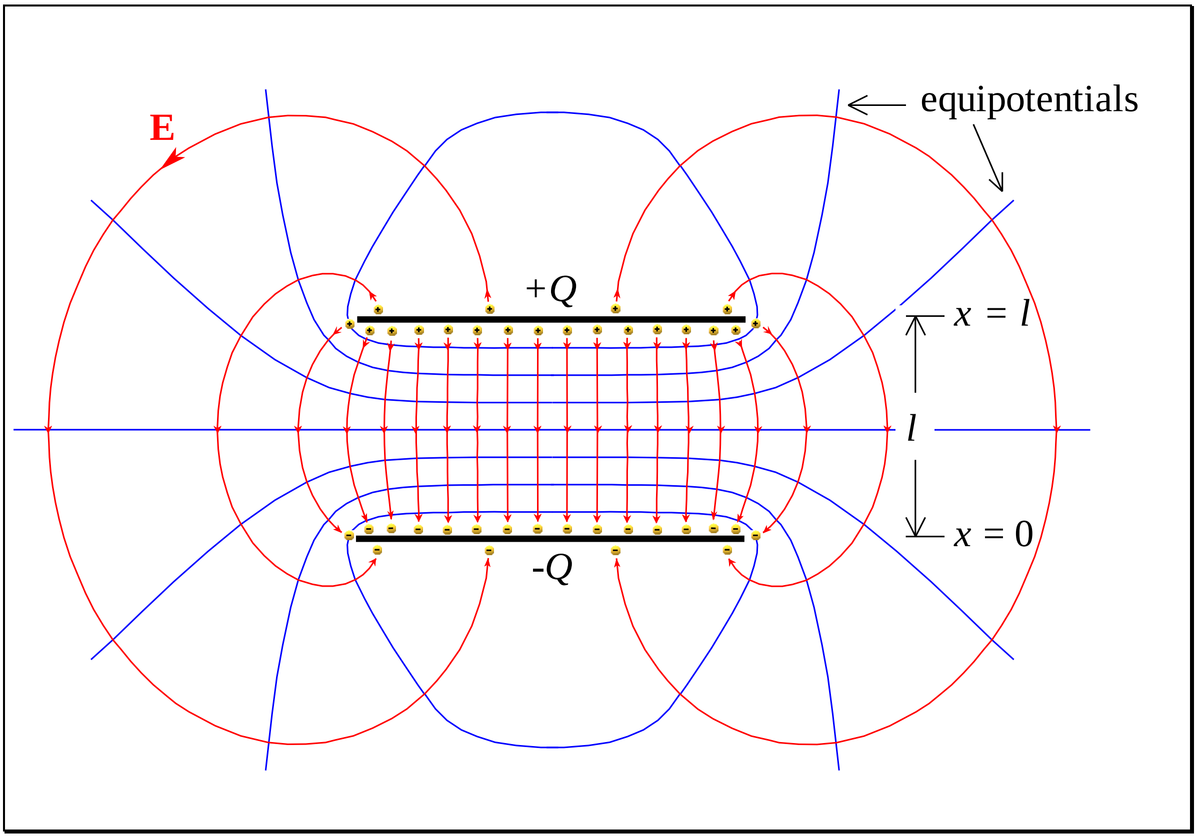


Figure 12.3 – Charge distribution and electric flux distribution between two large flat sheet conductors

The plate separation is l. The charge will distribute itself over the surface of each plate such that the tangential component of the electric flux density is zero. The attraction between charges of unlike sign will mean that most of the charge will be located on the surface of the plates opposite each other with some small proportion located on the surfaces remote from each other as shown.

Flux lines are assumed to start on positive charges and terminate on negative charges. Thus the majority of the electric flux lines will be located in the space between the plates, streaming directly between them. There will be a small fringing flux close to the edges, and a very small flux associated with the external surfaces of the plates.

The electric flux between the plates far from the edges will be approximately uniformly distributed so that we may say that the surface charge density on the inside of the plates can be approximately represented by a uniform charge density σ Cm^{-2}.

To determine the relationship between the surface charge density σ and **E**, assume a cylindrical Gaussian surface located at the top positively-charged plate as represented by the "pillbox" shown in the figure below:

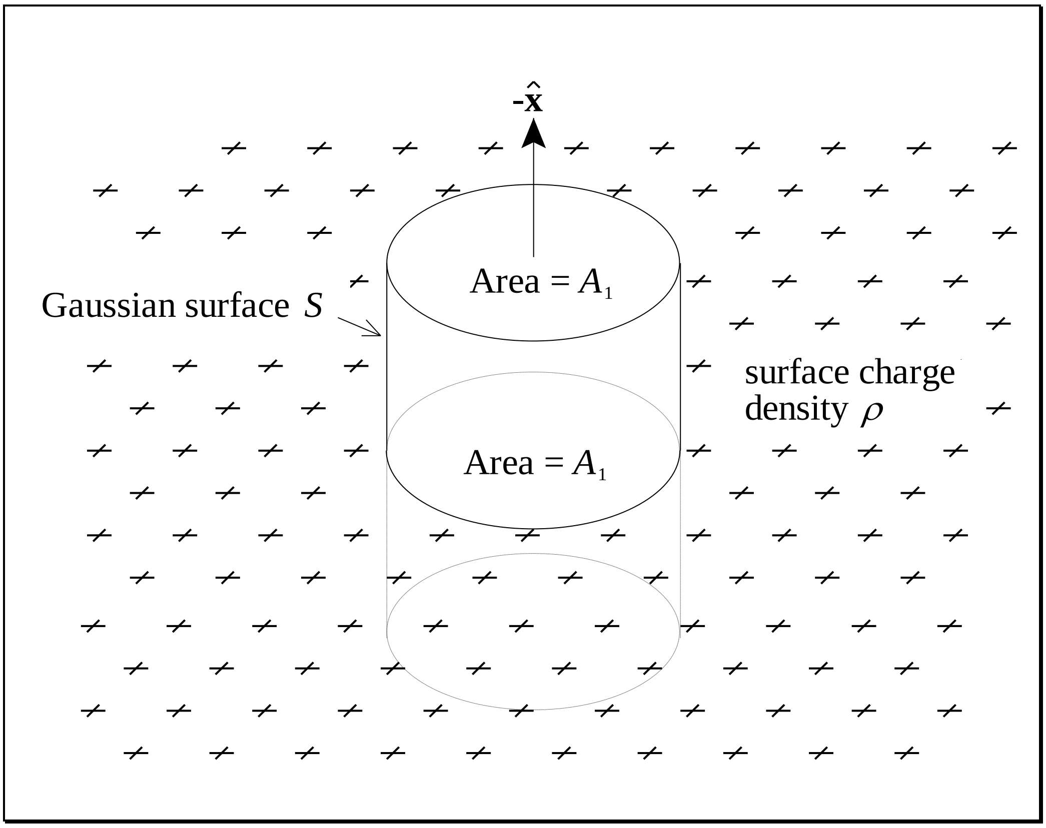


Figure 12.4 – A Gaussian surface (pillbox) intersecting a uniform surface charge density

The pillbox is a right cylinder and it extends through the surface charge distribution into the field-free region within the conductor (remember $\mathbf{D} = \mathbf{E} = \mathbf{0}$ within the conductor). The pillbox is bisected by the plane of uniform surface charge density.

The charge enclosed by the Gaussian surface *S* will be ρA_1 where A_1 is the area of the end of the "pillbox". The direction of the electric flux density vector **D** is perpendicular to the surface (why?), therefore **D** will not pass through the sides of the "pillbox". From Gauss' Law, the electric flux leaving the Gaussian surface *S* is:

$$\Psi = \oint \mathbf{D} \cdot \mathbf{dA} = q_{\text{enclosed}} \tag{12.2}$$

Now the charge enclosed by the Gaussian surface is just the amount of charge “captured” by the pillbox:

$$q_{\text{enclosed}} = \rho A_1 \tag{12.3}$$

The “closed” integral of the flux density over the entire Gaussian surface is composed of 3 parts: the top of the pillbox (area A_1), the side (cylinder) of the pillbox (area A_2), and the bottom of the pillbox (area A_3):

$$\oint \mathbf{D} \cdot \mathbf{dA} = \int_{A_1} \mathbf{D} \cdot \mathbf{dA}_1 + \int_{A_2} \mathbf{D} \cdot \mathbf{dA}_2 + \int_{A_3} \mathbf{D} \cdot \mathbf{dA}_3 \tag{12.4}$$

We already know that there is no flux streaming into the conductor ($\mathbf{D} = \mathbf{0}$ within the conductor) and so $\int_{A_3} \mathbf{D} \cdot \mathbf{dA}_3 = 0$. Also, since the direction of **D** is parallel to the sides of the pillbox, then we must have $\int_{A_2} \mathbf{D} \cdot \mathbf{dA}_2 = \int_{A_2} D dA_2 \cos 90° = 0$.

This leaves just the component due to the top of the pillbox and we have:

$$\oint \mathbf{D} \cdot \mathbf{dA} = \int_{A_1} \mathbf{D} \cdot \mathbf{dA}_1 = \int_{A_1} D \cdot dA_1 = D \int_{A_1} dA_1 = DA_1 \tag{12.5}$$

Therefore, Gauss' Law gives us:

$$DA_1 = \rho A_1 \tag{12.6}$$

and so the magnitude of the flux density is just:

$$D = \rho \tag{12.7}$$

We can now state the vector flux density as:

$$\mathbf{D} = -\rho \hat{\mathbf{x}} \tag{12.8}$$

and therefore the electric field intensity is:

$$\mathbf{E} = -\frac{\rho}{\varepsilon_0} \hat{\mathbf{x}} \tag{12.9}$$

Now if we assume that we can neglect the areas at the edge of the parallel plate structure where the field will fringe slightly, then the surface charge density σ is approximated as:

$$\rho \approx \frac{Q}{A} \tag{12.10}$$

where A is the total area of the inside surface of the parallel-plate structure. Therefore, the electric field between the plates can be approximated as:

$$\mathbf{E} = -\frac{Q}{\varepsilon_0 A} \hat{\mathbf{x}} \tag{12.11}$$

Note that the negative sign is simply due to the way we defined the *x*-direction (from negative plate to positive plate). Therefore, as the field plot in Figure 12.3 shows, the electric field points from the positively-charged plate to the negatively-charged plate.

We saw previously (Topic 11.5) that the potential difference, in moving from the negatively-charged plate along a uniform (constant magnitude) electric field line to the positively-charged plate is given by:

$$V = El \tag{12.12}$$

where the polarity of V is such that (+) is on the positively-charge plate and (-) is on the negatively-charged plate.

Using Eq. (12.11), the voltage is then:

$$V = \frac{Ql}{\varepsilon_0 A} \tag{12.13}$$

From the definition of capacitance, $C = q/v$, we then have a formula for the capacitance between two large, closely-spaced, parallel conducting plates in air:

$$C = \frac{\varepsilon_0 A}{l} \tag{12.14}$$

The capacitance of a parallel plate capacitor

where A is the area of either of the two parallel plates, and l is the distance between them.

12.3 Capacitances in Series and Parallel

12.3.1 Capacitances in Series

Assume that two initially uncharged metal capacitor structures are connected in series as shown in the figure below:

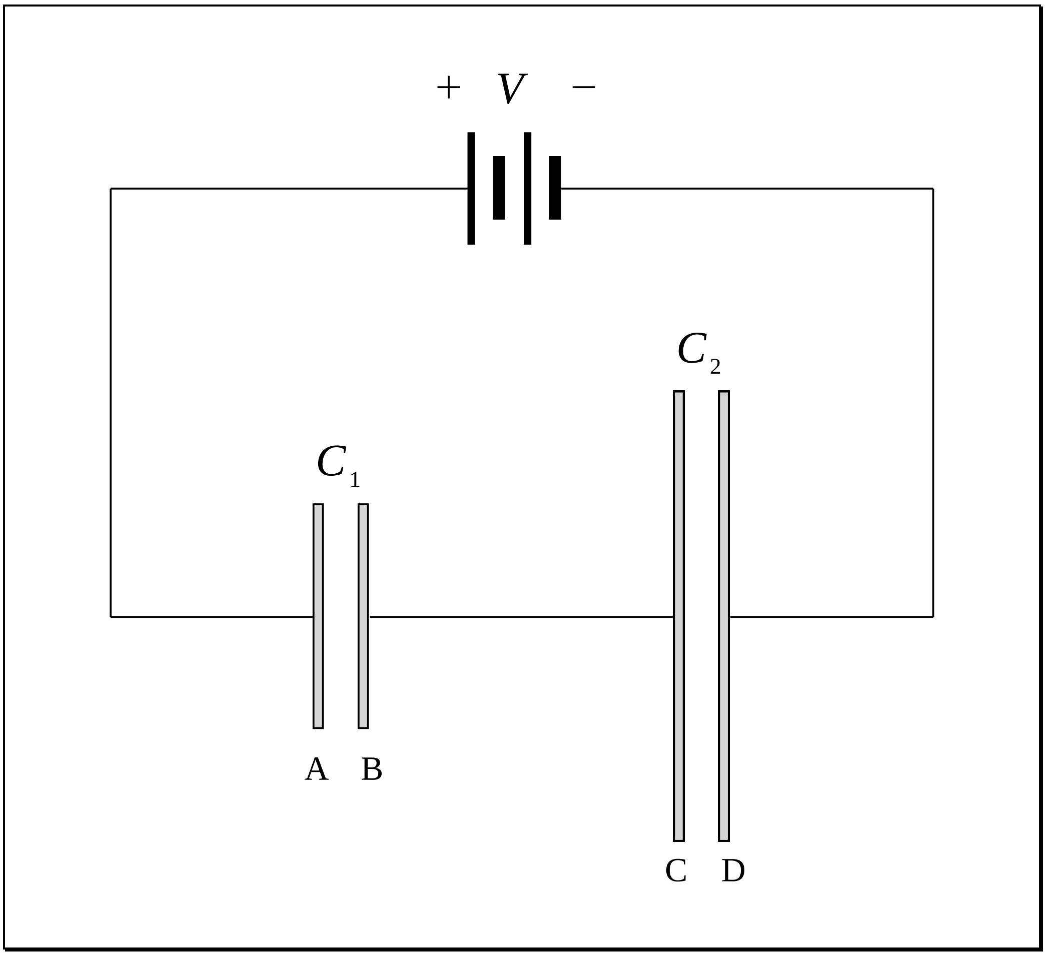


Figure 12.5 – A voltage source *V* with two series-connected capacitors

The interconnecting link between the structures may be assumed sufficiently long that capacitances between plates A & C or A & D etc., might be assumed negligible. If a source of potential difference such as a battery is now connected between conductors A and D as shown, electrons will be removed from plate A producing a positive charge on plate A and a similar number of electrons (although not the same electrons) will be deposited on plate D producing a negative charge on plate D. The total charge on the structure made up of the two interconnected conducting plates B & C will not change since there is no conducting path to the outside world along which electrons can flow. The conduction electrons within a conductor are relatively loosely bound to their parent nuclei and to all intents and purposes able to be considered as free charges. These charges are however "trapped" within this structure and will tend to re-arrange themselves under the influence of the electric fields produced by the positive charge deposited on conducting plate A and the negative charge deposited on conducting plate B.

The positive charge (excess protons) on plate A will attract an excess of electrons onto plate B while the negative charge (excess electrons) on plate D will repel electrons from plate C and produce a net positive charge (protons) on plate C. The induced charges on plates B & C will be equal in magnitude. They will also be equal in magnitude to the charges deposited on plates A & D by the battery since electric flux must terminate on positive or negative charge. We therefore have the charge distribution in the figure below together with potential differences as shown:

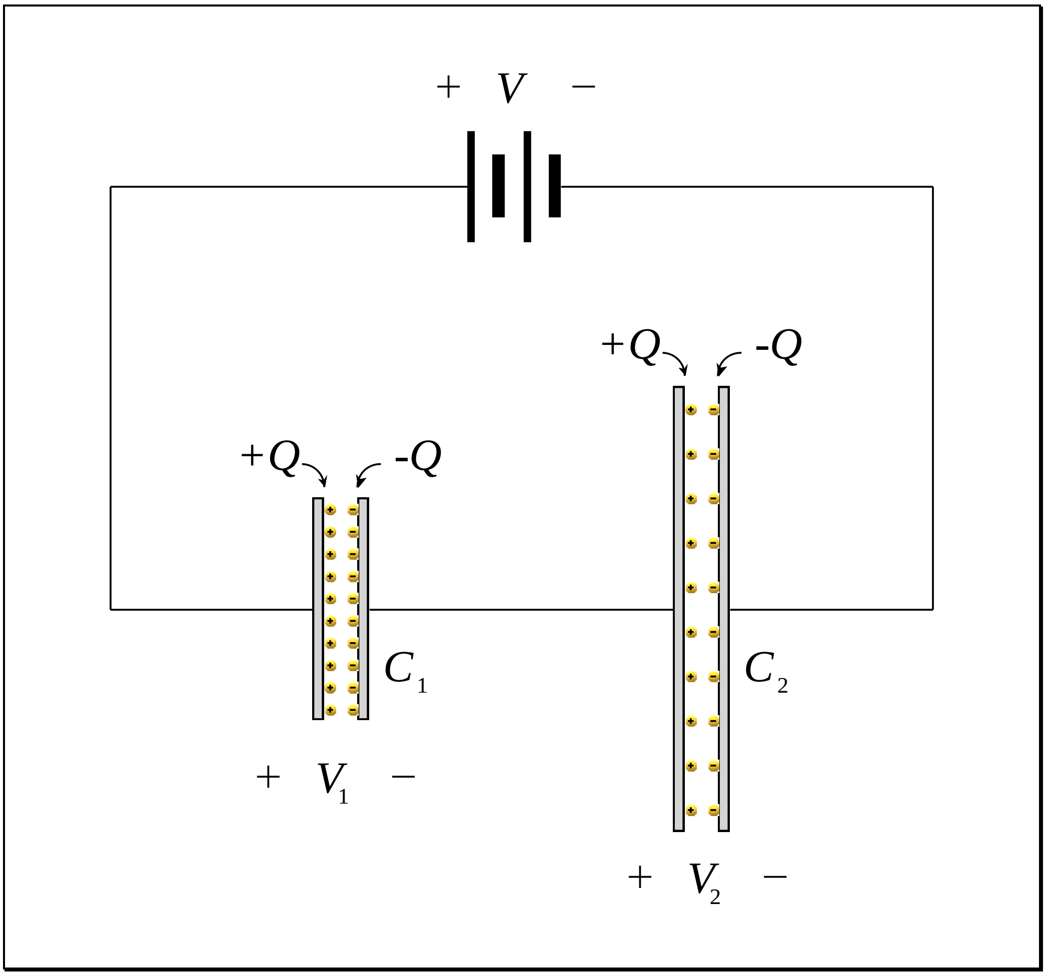


Figure 12.6 – Charge and voltages on series-connected capacitors

The voltages which result from the presence of the charges on the structure are such that KVL gives:

$$V = V_1 + V_2 \tag{12.15}$$

But from the definition of capacitance, we have:

$$C_1 = \frac{Q}{V_1} \quad \text{and} \quad C_2 = \frac{Q}{V_2} \tag{12.16}$$

Using these formula in Eq. (12.15), we have:

$$V = \frac{Q}{C_1} + \frac{Q}{C_2} = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{Q}{C_{\text{ser}}} \tag{12.17}$$

Therefore, the capacitance of two series-connected capacitors is:

$$\boxed{\frac{1}{C_{\text{ser}}} = \frac{1}{C_1} + \frac{1}{C_2}} \tag{12.18}$$

12.3.1 Capacitances in Parallel

Assume that two initially uncharged metal capacitor structures are connected in parallel with a battery as shown in the figure below:

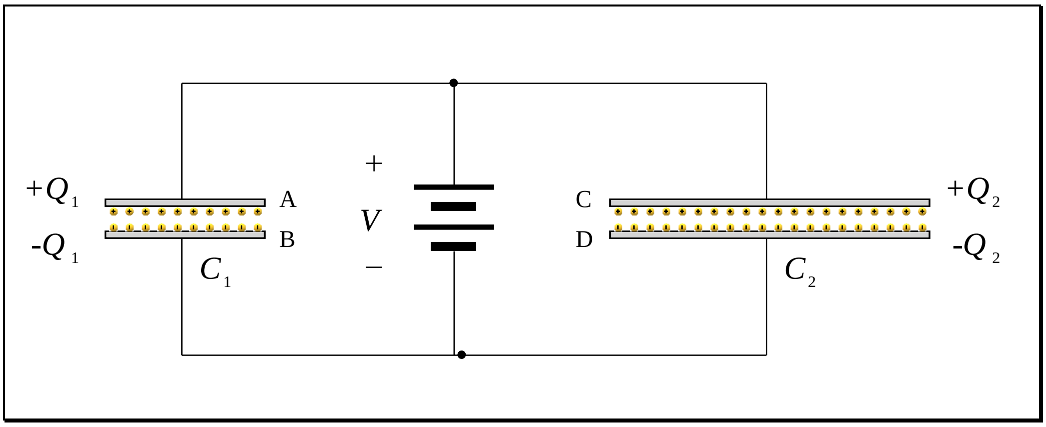


Figure 12.7 – Capacitor structures connected in parallel

Upon connection of the battery of potential difference V to the parallel-connected structure, charge will distribute itself between the capacitors C_1 and C_2, such that:

$$Q_1 = C_1 V \quad \text{and} \quad Q_2 = C_2 V \tag{12.19}$$

From the definition of capacitance, the effective capacitance of the parallel-connected structure C_{par} is given by:

$$C_{\text{par}} = \frac{Q_{total}}{V} = \frac{Q_1 + Q_2}{V} = \frac{Q_1}{V} + \frac{Q_2}{V} \tag{12.20}$$

and so, using Eqs. (12.19), the capacitance of two parallel-connected capacitors is:

$$C_{\text{par}} = C_1 + C_2 \tag{12.21}$$

In summary, capacitors in series and parallel are treated like conductances.

12.4 Dielectrics

In the majority of capacitors the material occupying the space between the plates is not air or vacuum but some form of insulating or dielectric material. It is used to provide physical support for each plate relative to the other and to enhance certain electrical characteristics as we will see below.

In a dielectric, electrons are very firmly bound to their parent nuclei and are not free to wander as they are in a conductor. However, under the influence of an applied electric field there will be a tendency for the electron cloud of a given atom to be displaced *in opposition* to the direction of the field, thus exposing slightly a net positive charge from the nucleus. The displaced cloud of electrons will appear as a net negative charge and so the overall atom will appear as a small electric dipole.

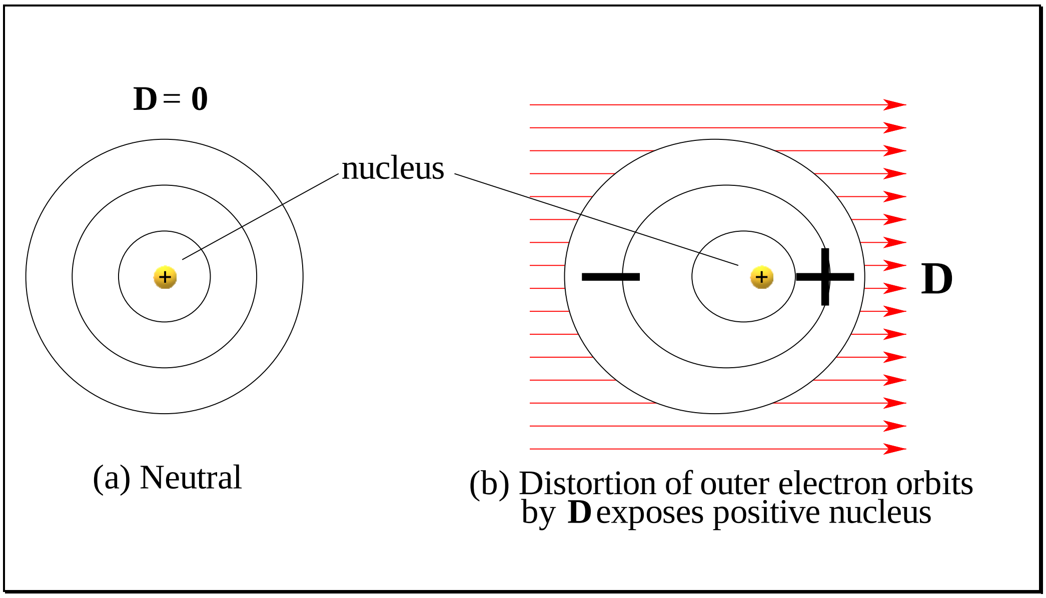


Figure 12.8 – The effect of electric flux density D on an atom

Within the dielectric lattice the atoms overlap and hence positive and negative charges of the dipoles neutralise each other. There will therefore be no noticeable effect within the dielectric. But on the upper and lower surfaces as in the figure below, there will be no neutralisation, so that the dielectric may be represented as a material with a layer of bound charge on its upper and lower surfaces.

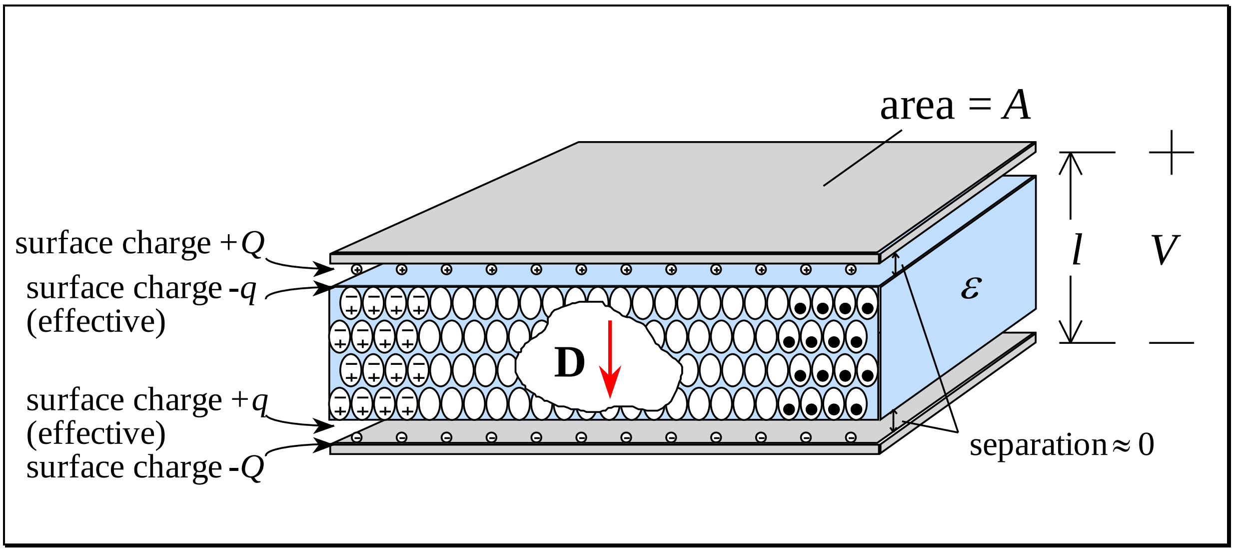


Figure 12.9 – The effect of a dielectric inserted between the plates of a parallel-plate capacitor

The charges +Q and -Q on the inside surfaces of the metal plates are unbound or free charges.

The electric field within the dielectric of Figure 12.9 is due to charge layers Q and -q. Each charge layer will be uniformly distributed over the surface of the capacitor so that two opposing fields will be set up – one due to the applied charge Q and one due to the induced charge -q. Each charge layer may be assumed to contribute to a *local* value of electric field intensity within the dielectric such that:

$$E = \frac{Q}{\varepsilon_0 A} - \frac{q}{\varepsilon_0 A} = \frac{Q - q}{\varepsilon_0 A} \tag{12.22}$$

and with **E** directed from the upper to the lower plate.

The potential difference between the plates is given by:

$$V = \frac{(Q-q)l}{\varepsilon_0 A} \tag{12.23}$$

where l is the distance between the plates as shown. The capacitance of the structure, neglecting field fringing at the edges will be given by:

$$C = \frac{Q}{V} = \frac{Q}{Q-q}\frac{\varepsilon_0 A}{l} \tag{12.24}$$

If we compare Eqs. (12.14) and (12.24) we see that for structures with similar surface areas A and plate separations l, the capacitance is increased by the presence of the dielectric in the ratio:

Relative permittivity defined

$$\varepsilon_r = \frac{Q}{Q-q} \quad \text{where} \quad \varepsilon_r \geq 1 \tag{12.25}$$

The *relative permittivity*, ε_r, is a constant of the dielectric material between the plates.

The capacitance of a parallel-plate capacitor can now be written as:

The capacitance of a parallel plate capacitor

$$C = \frac{\varepsilon_r \varepsilon_0 A}{l} = \frac{\varepsilon A}{l} \tag{12.26}$$

where the *permittivity*, ε, is:

$$\varepsilon = \varepsilon_r \varepsilon_0 \tag{12.27}$$

12.5 Electric Flux Density D and Electric Field Intensity E within a Dielectric

In free space the relationship between the electric flux density **D** and the electric field intensity **E** is written as:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} \tag{12.28}$$

The electric flux density **D** is defined in terms of flux streaming from essentially isolated or "free" charges that have been externally applied to "the system". In the case discussed above involving the parallel-plate capacitor containing the dielectric of relative permittivity ε_r, the externally applied free charge is simply Q. Charge q is considered as bound charge and hence to be different in nature from the free charge Q. In this case, where the field is uniform, the magnitude of the electric flux density then becomes simply:

$$\mathbf{D} = -\frac{Q}{A}\hat{\mathbf{x}} \tag{12.29}$$

In other words, **D** takes no account of the induced charge q and the presence of the dielectric does not affect the value of **D** for a given applied free charge Q.

However, the electric field intensity **E** within the dielectric is directly related to forces on charges and potential gradient. It will therefore include the effects of the induced charges q, since their presence will directly affect the local value of the electric potential gradient and the force on any electron situated within the dielectric.

The concept of free and bound charges in any system containing a dielectric is extremely important in the understanding of these systems. The issue may sometimes be clouded in any system where the applied free charge is not held constant. This is the case for example when the applied *potential difference* between the plates of a capacitor is held constant while some other parameter of the system is changed.

If a uniform electric field distribution is assumed within the dielectric of the capacitor of Figure 12.9, and if we neglect the effects of fringing near the edges of the structure, the electric field intensity **E** within the dielectric will be given by the negative of the gradient of the potential with distance across the dielectric. Therefore, from Eqs. (12.23) and (12.25):

$$\begin{aligned}\mathbf{E} &= -\frac{(Q-q)}{\varepsilon_0 A}\hat{\mathbf{x}} \\ &= -\frac{Q}{\varepsilon_r \varepsilon_0 A}\hat{\mathbf{x}}\end{aligned} \tag{12.30}$$

Comparing Eqs.(12.29) and (12.30) it can be seen that within the dielectric:

$$\mathbf{D} = \varepsilon_r \varepsilon_0 \mathbf{E} = \varepsilon \mathbf{E} \tag{12.31}$$

In other words, for a given charge density on the surface of the metallic plates, the effect of the dielectric is to reduce the value of the electric field intensity below that which would have occurred had the dielectric not been present.

12.6 Capacitor *v*-*i* Relationships

We now seek a *v*-*i* relationship for the capacitor. From the definition of current:

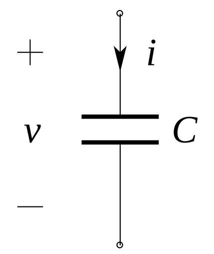
$$i = \frac{dq}{dt} \tag{12.32}$$

we substitute $q = Cv$ and obtain:

$$i = C\frac{dv}{dt} \tag{12.33}$$

The capacitor's *v*-*i* branch derivative relationship

The circuit symbol for the capacitor is based on the construction of the physical device, and is shown below together with the passive sign convention for the voltage and current:



The circuit symbol for the capacitor

Figure 12.10

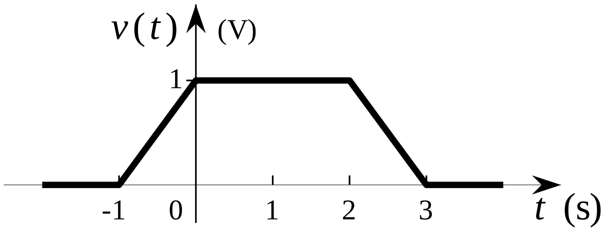
The capacitor voltage may be expressed in terms of the current by integrating Eq. (12.33):

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i\,d\tau + v(t_0) \tag{12.34}$$

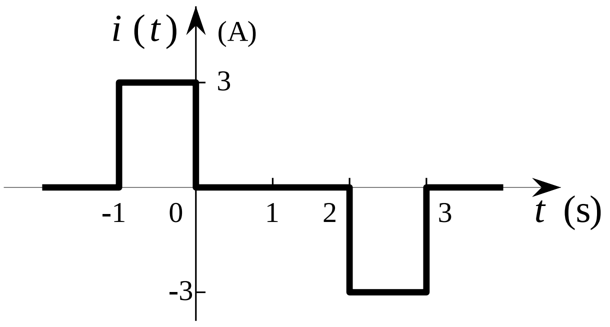
The capacitor's *v*-*i* branch integral relationship

EXAMPLE 12.1 Capacitor Voltage and Current Waveforms

The voltage waveform across a 3 F capacitor is shown below:



Since the voltage is zero and constant for $t < -1$, the current is zero in this interval. The voltage then begins to increase at the linear rate $dv/dt = 1\,\mathrm{Vs}^{-1}$, and thus a constant current of $i = C\,dv/dt = 3\,\mathrm{A}$ is produced. During the following 2 second interval, the voltage is constant and the current is therefore zero. The final decrease of the voltage causes a negative 3 A and no response thereafter. The current waveform is sketched below:



12.7 Energy Stored in a Capacitor

The power delivered to a capacitor is:

$$p = vi = Cv\frac{dv}{dt} \tag{12.35}$$

and the energy stored in its electric field is therefore:

$$\begin{aligned} w_C(t) &= \int_{t_0}^{t} p\,dt + w_C(t_0) \\ &= C\int_{t_0}^{t} v\frac{dv}{dt}dt + w_C(t_0) \\ &= C\int_{v(t_0)}^{v(t)} v\,dv + w_C(t_0) \\ &= \frac{1}{2}C\left[v^2(t) - v^2(t_0)\right] + w_C(t_0) \end{aligned} \tag{12.36}$$

If the capacitor voltage is zero at t_0, then the electric field, and hence the stored capacitor energy, $w_C(t_0)$, is also zero at that instant. We then have:

$$w_C(t) = \frac{1}{2}Cv^2(t) \tag{12.37}$$

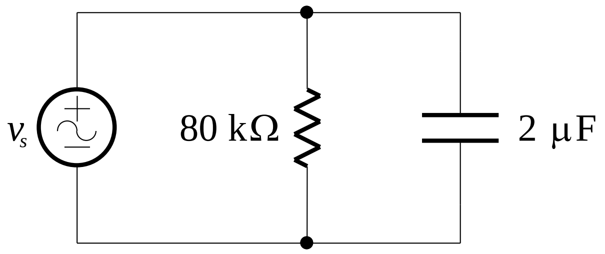
The stored energy in a capacitor

We can see that the energy stored in a capacitor depends only on the capacitance and the voltage. Therefore, a finite amount of energy can be stored in a capacitor even if the current through the capacitor is zero.

Whenever the voltage is not zero, and regardless of its polarity, energy is stored in the capacitor. It follows, therefore, that power must be delivered to the capacitor for a part of the time and recovered from the capacitor later.

EXAMPLE 12.2 Energy Stored in a Capacitor

A sinusoidal voltage source is connected in parallel with an $80\ \text{k}\Omega$ resistor and a 2 μF capacitor, as shown below:



The $80\ \text{k}\Omega$ resistor in the circuit represents the dielectric losses present in a typical $2\ \mu\text{F}$ capacitor. Let $v_s = 325\sin(100\pi t)\ \text{V}$, which corresponds to a typical low voltage 230 V RMS household supply.

The current through the resistor is:

$$i_R = \frac{v}{R} = 4.063\sin(100\pi t)\ \ \text{mA}$$

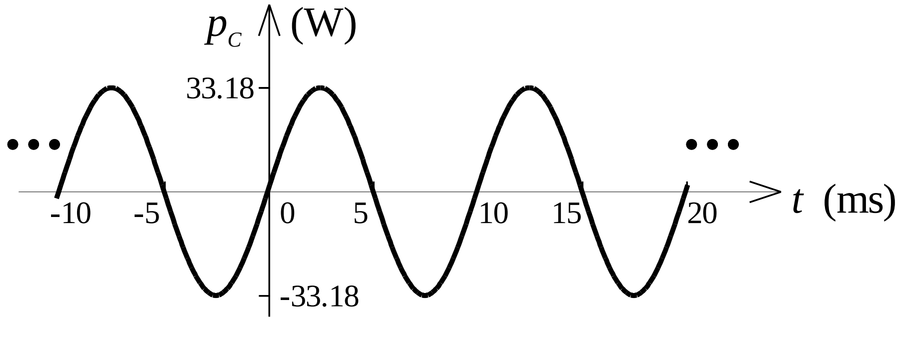
and the current through the capacitor is:

$$i_C = C\frac{dv}{dt} = 2\times10^{-6}\frac{d}{dt}\left[325\sin(100\pi t)\right] = 204.2\cos(100\pi t)\ \ \text{mA}$$

The power delivered to the capacitor is:

$$p_C = vi_C = 325\sin(100\pi t)\times 0.2042\cos(100\pi t) = 33.18\sin(200\pi t)\ \ \text{W}$$

A graph of the power delivered to the capacitor versus time is shown below:

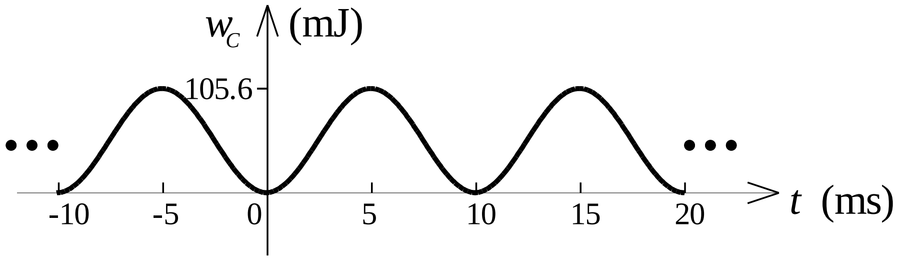


The *instantaneous* power delivered to the capacitor is sinusoidal, and varies at *twice* the frequency of the voltage source. Negative power *delivered to* the capacitor means that power is *sourced from* the capacitor. Note that the *average* power dissipated in the capacitor is zero.

The energy stored in the capacitor is:

$$w_C = \frac{1}{2}Cv^2 = \frac{1}{2} \times 2 \times 10^{-6}[325 \sin(100\pi t)]^2 = 105.6 \sin^2(100\pi t) \quad \text{mJ}$$

A graph of the stored energy versus time is shown below:



The energy stored is sinusoidal and varies at twice the frequency of the voltage source, but it also has a finite *average* component of 52.81 mJ.

We see that the energy increases from zero at $t = 0$ to a maximum of 105.6 mJ at $t = 5\,\text{ms}$ and then decreases to zero in another 5 ms. During this 10 ms interval, the energy dissipated in the resistor as heat is:

$$w_R = \int_0^{0.01} p_R dt = \int_0^{0.01} 1.320 \sin^2(100\pi t) dt = \int_0^{0.01} 0.6602[1 - \cos(200\pi t)]dt = 6.602 \quad \text{mJ}$$

Thus, an energy equal to 6.25% of the maximum stored energy is lost as heat in the process of storing and removing the energy in the physical capacitor. Later we will formalise this concept by defining a quality factor Q that is proportional to the ratio of the maximum energy stored to the energy lost per period.

12.8 Summary of Important Capacitor Characteristics

Several important characteristics of the ideal capacitor are summarised below.

The capacitor is an open-circuit to DC

1. If a constant voltage is held across a capacitor, then $dv/dt = 0$ and subsequently no current enters (or leaves) it. A capacitor is thus **an open-circuit to DC**. This fact is represented by the capacitor symbol.

A capacitor voltage cannot change instantaneously

2. A capacitor **voltage cannot change instantaneously**, for this implies $dv/dt = \infty$, and the capacitor would require infinite current. Thus, capacitor voltage is smooth and continuous. This fact will be used frequently when undertaking *transient analysis* of circuits.

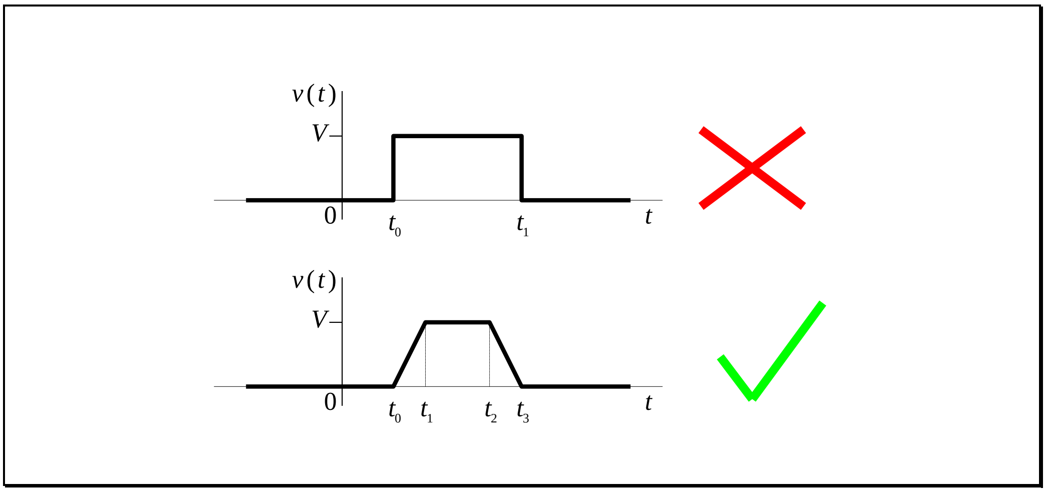


Figure 12.11

3. For an ideal capacitor, **current can change instantaneously**.

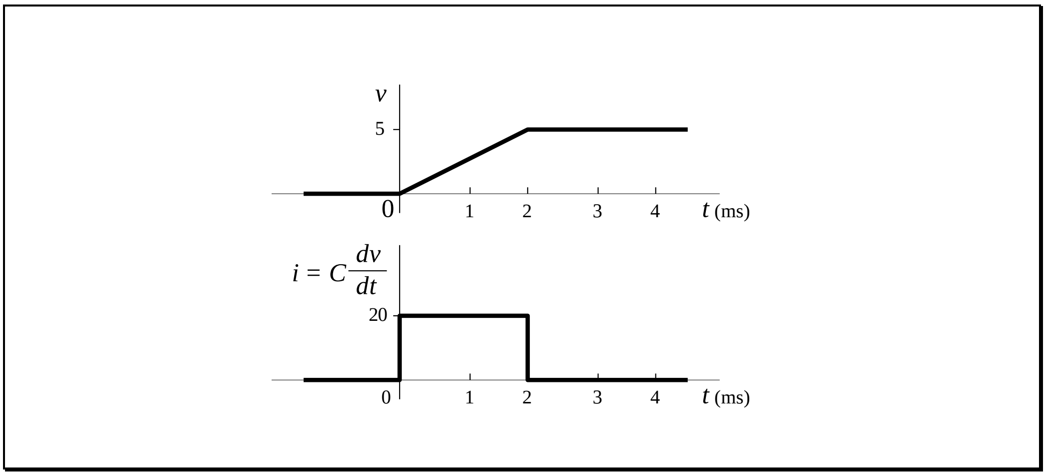


Figure 12.12

4. An ideal capacitor never dissipates energy, but stores it and releases it using its electric field.

12.9 Practical Capacitors

Practical capacitors are manufactured for different values, voltage and current ratings, accuracy, volumetric efficiency, temperature stability, etc. As such, their construction will play a role in their electrical behaviour. To represent these physical components in a circuit we need to model their non-ideal and parasitic effects. There are many different types of capacitor construction. Some are shown below, labelled by the type of dielectric:

Some types of capacitors

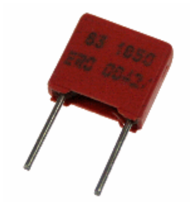
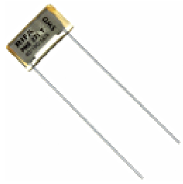
 polystyrene	 polypropylene	 polycarbonate
 PTFE (Teflon)	 polyester	 paper
 ceramic	 tantalum electrolytic	 aluminium electrolytic

Figure 12.13 – Some types of capacitors

A reasonably accurate circuit model of a real capacitor is shown below:

Model of a real capacitor

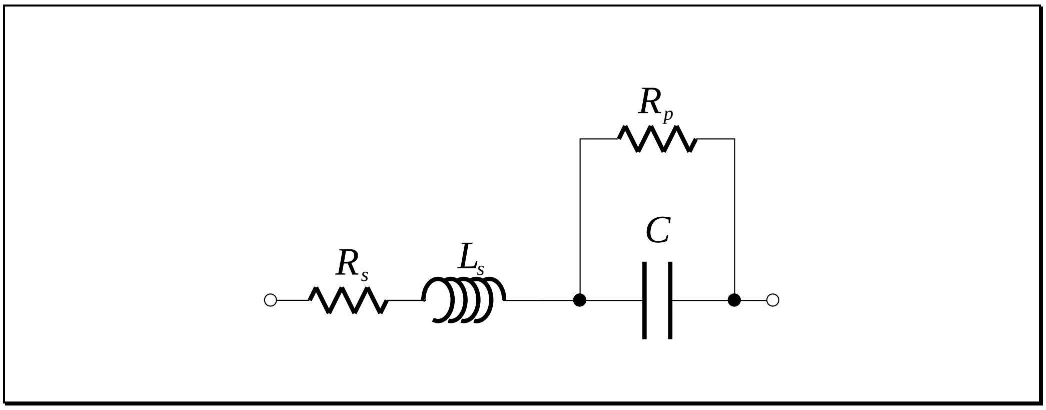


Figure 12.14

In the model, the series resistance R_s takes into account the finite resistance of the plates used to make the capacitor. The series inductance L_s is used to model the fact that a current is required to charge and discharge the plates, and this current must have a magnetic field. Finally, no practical material is a perfect insulator, and the resistance R_p represents conduction through the dielectric.

Parasitic elements defined

We call R_s, L_s and R_p *parasitic elements*. Capacitors are designed to minimize the effects of the parasitic elements consistent with other requirements such as physical size and voltage rating. However, parasitics are always present, and when designing circuits care must be taken to select components for which the parasitic effects do not compromise the proper operation of the circuit.

12.9.1 Electrolytic Capacitors

Electrolytic capacitors have a large capacitance per unit volume, but are polarised

One side (the anode) of an electrolytic capacitor is formed from a foil of aluminium which is anodised to produce an oxide layer which is the dielectric. The oxide-coated foil is immersed in an electrolytic solution which forms the cathode. A large capacitance per unit volume is achieved. However, exposure to a reverse voltage for a long time leads to rapid heating of the capacitor and to breakdown. Thus, electrolytic capacitors are polarised and it must be ensured that the correct voltage polarity is applied to avoid failure.

12.10 Circuit Analysis with Capacitors

For the ideal capacitor, all of the relationships between voltage and current are linear relationships (apart from the integral relationships that have an initial condition term). Consequently, circuit analysis techniques which rely on the linearity property (such as nodal analysis, mesh analysis, superposition, Thévenin's theorem and Norton's theorem) can be applied to circuits containing capacitors.

Linear circuit analysis techniques can be applied to circuits with capacitors

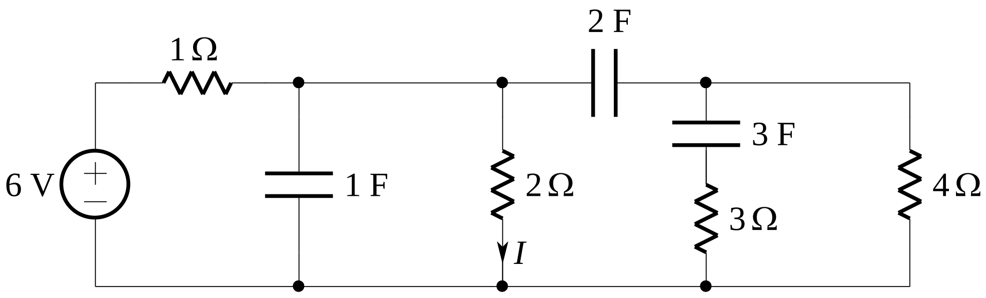
12.10.1 DC Circuits

An ideal capacitor behaves as an open-circuit to DC. We can use this fact to determine *steady-state* voltages and currents in DC circuits that contain capacitors. Other sources, such as sinusoidal sources, will be treated later.

A DC circuit is treated as a purely resistive circuit in the "steady-state"

EXAMPLE 12.3 DC Analysis of a Circuit with Capacitors

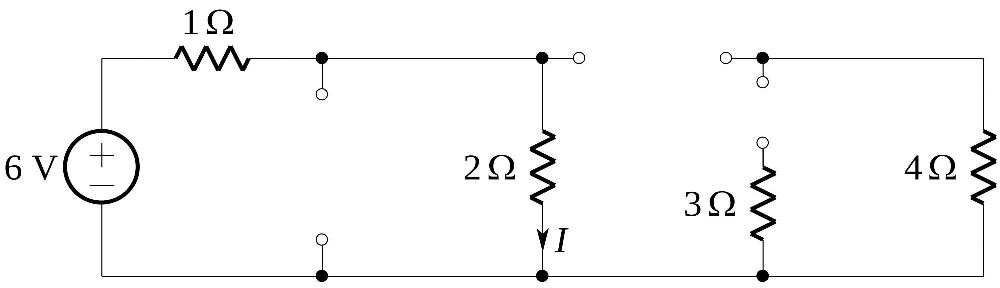
Determine the steady-state direct current I in the circuit below:



The circuit has one independent voltage source whose value is constant. For a resistive circuit we would naturally anticipate that all voltages and currents are constant. However, this is not a resistive circuit. Yet, our intuition suggests that the constant-valued voltage source produces constant-valued responses. This fact will be confirmed more rigorously later. In the meantime, we shall use the result that a circuit containing only constant-valued sources, in the steady-state, is a DC circuit.

12.28

Since for DC all capacitors behave like open-circuits, we can replace the original circuit with an equivalent resistive circuit:



The steady-state direct current is now given by Ohm's Law:

$$I = \frac{6}{1+2} = 2\text{ A}$$

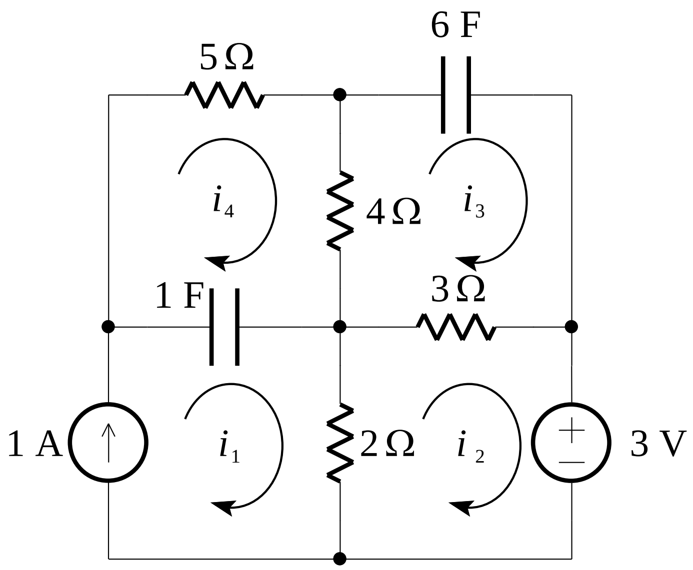
12.10.2 Nodal and Mesh Analysis

Just as we analysed resistive circuits with the use of node and mesh equations, we can write a set of equations for circuits that contain capacitors in addition to resistors and sources. The procedure is similar to that described for the resistive case – the difference being that for capacitors the appropriate relationship between voltage and current is used in place of Ohm's Law.

Nodal and mesh analysis can be applied to circuits with capacitors

EXAMPLE 12.4 Mesh Analysis of a Circuit with Capacitors

Consider the circuit shown below:



By mesh analysis, for mesh i_1:

$$i_1 = 1$$

For mesh i_2:

$$2(i_2 - i_1) + 3(i_2 - i_3) = -3$$

For mesh i_3:

$$3(i_3 - i_2) + 4(i_3 - i_4) + \frac{1}{6}\int_{-\infty}^{t} i_3 dt = 0$$

For mesh i_4:

$$\frac{1}{1}\int_{-\infty}^{t} (i_4 - i_1) dt + 5i_4 + 4(i_4 - i_3) = 0$$

The equations obtained for meshes i_3 and i_4 are *integral equations* since they contain integrals.

Writing the equations for a circuit, as in the preceding example, is not difficult. Finding the solution of equations like these, however, is another matter – it is no simple task. Thus, with the exception of some very simple circuits, we shall have to resort to additional concepts and techniques to be introduced later.

12.11 Summary

- Capacitance is defined as the ratio of charge to voltage of a structure:

$$q = Cv$$

- The capacitance of a thin parallel-plate capacitor (no fringing effects) is:

$$C = \frac{\varepsilon A}{l} = \frac{\varepsilon_r \varepsilon_0 A}{l}$$

where A is the area of the plates, l is the separation between them, and the relative permittivity ε_r is a property of the insulating material (dielectric).

- Capacitors in series and parallel are combined in the same way as are conductances.

- The fundamental relationship between electric flux density and electric field intensity is:

$$\mathbf{D} = \varepsilon \mathbf{E}$$

- The *v*-*i* relationship for a capacitor is:

$$i = C\frac{dv}{dt}$$

- A capacitor behaves like an **open-circuit** to **direct current** in the steady-state.

- The voltage across a capacitor cannot change instantaneously.

- The energy stored in a capacitor is:

$$w_C = \frac{1}{2}Cv^2$$

- Writing node and mesh equations for circuits containing capacitors is done in the same manner as for resistive circuits. Obtaining solutions of equations in this form will be avoided, except for simple circuits.

12.12 References

Beard, G.: *Electrostatics and Magnetostatics*, NSWIT, 1985.

Exercises

1.

A conductor of area A carrying a charge Q is situated parallel to and distance d from an earthed plane conductor of the same shape. Derive an expression for the capacitance of the two plates. Neglect fringing.

2.

A voltage of 6 kV is maintained between the plates of a parallel plate capacitor. The area of each plate is 10^4 mm^2 and they are separated by 6 mm. Calculate the field strength and flux density between the plates, and the capacitance:

(a) if the separating medium is air.

(b) if an epoxy resin dielectric of relative permittivity 5 and thickness 6 mm is placed between the plates.

(c) if the dielectric thickness is only 5 mm (calculate the field strength in the air and the dielectric). Could you recommend this arrangement if the breakdown strength of air is 3×10^6 Vm^{-1}?

3.

(a) Derive an expression for the capacitance per unit length of a co-axial cable. Let the radii of the inner and outer conductors be r_1 and r_2, and let the insulation between them have relative permittivity ε_r.

(b) If $r_1 = 1\,\text{mm}$, $r_2 = 9\,\text{mm}$ and $\varepsilon_r = 4.0$ (oil impregnated paper), calculate the capacitance of 10 km of the cable. If the electric field strength in the insulation is not to exceed 20 kV/mm, find the maximum voltage which can be applied to the cable. Where is the insulation likely to begin to break down?

4.

Using the definitions of potential difference and capacitance, show that the energy stored in a capacitor is given by:

$$w_C = \frac{q^2}{2C} = \frac{1}{2}Cv^2 = \frac{1}{2}qv$$

5.

A parallel-plate capacitor is charged and then isolated from the supply. The plates are then moved further apart. Describe the changes that occur in the potential difference between the plates, the energy stored in the capacitor, the electric field strength between the plates, the electric flux density and the capacitance.

6.

A parallel-plate capacitor is maintained at constant voltage. The plates are then moved further apart. Describe the changes that occur in the charge stored on the plates, the energy stored by the capacitor, the electric field strength between the plates, the electric flux density and the capacitance.

7.

An air-spaced variable capacitor set to 1.5 nF is charged to a potential difference of 500 V and then disconnected from the source. The separation of the plates is then changed, so that the capacitance is 500 pF. Determine:

(a) the energy stored in the capacitor in the initial condition;

(b) the potential difference between the plates in the new position;

(c) the energy stored in the capacitor in the new position; and

(d) the mechanical work done on the electrostatic system during the change in plate separation.

8.

An air-spaced variable capacitor is set to a capacitance of 1 nF and is connected to a 400 V supply. Without disconnecting the supply the plates are allowed to move closer together so that capacitance is increased to 3 nF. Determine:

(a) the energy stored in the capacitor before and after the plates are moved;

(b) the charge on the plates before and after the movement;

(c) the energy supplied by the source; and

(d) the mechanical work done on the capacitor by the electrical supply.

9.

A parallel-plate capacitor has two plates each of area $5 \times 10^{-3}\ \text{m}^2$ separated by a 2 mm thick sheet of mica of relative permittivity 5. The capacitor is charged to a potential difference of 500 V. Determine:

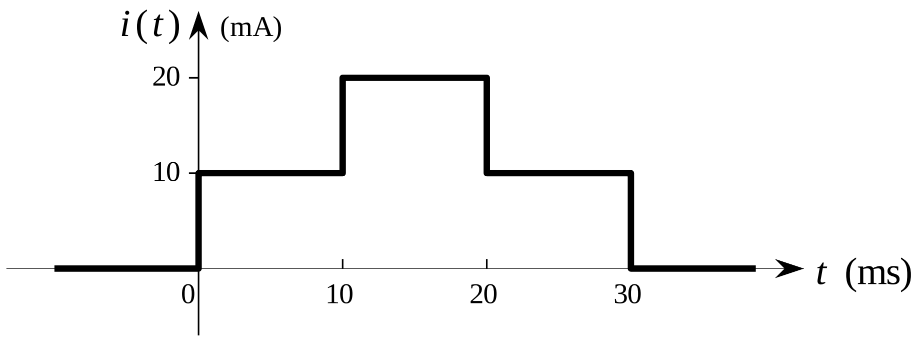
(a) the electric field strength in the mica;

(b) the flux density in the mica;

(c) the energy stored per cubic metre in the mica.

10.

A $25\ \mu\text{F}$ capacitor having no voltage across it at $t = 0$ is subjected to the single pulse of current shown below.



Determine the voltage across, the power entering, and the energy stored in C at $t =$:

(a) 17 ms

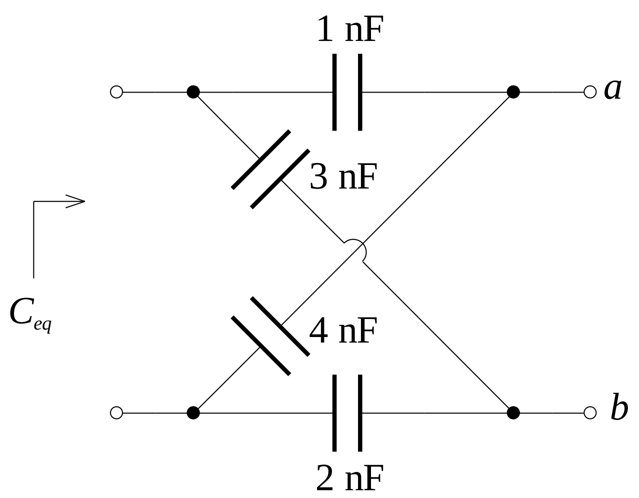
(b) 40 ms

11.

Find C_{eq} for the lattice network shown below if terminals a and b are:

(a) open-circuited as shown

(b) short-circuited



12.

The series combination of a $4\,\mu\text{F}$ and a $3\,\mu\text{F}$ capacitor is in series with the parallel combination of a $2\,\mu\text{F}$, a $1\,\mu\text{F}$ and a $C\,\mu\text{F}$ capacitor.

(a) What is the maximum possible value for the equivalent capacitance of the five capacitors?

(b) Repeat for the minimum value.

(c) Find C if $C_{eq} = 1.5\,\mu\text{F}$.