13 The Magnetic Field

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13.2

Introduction

When a charge is at rest, it gives rise to an electric field. When a charge moves at a uniform velocity, it gives rise to a magnetic field. Since a current, by definition, is the movement of charge, a current has an associated magnetic field.

We will formulate the concept of a magnetic field, and quantify some of the terms used in dealing with magnetic fields, in a manner similar to that for electrostatics. We will extend the model of "flux" to the magnetic realm, and use the same concepts as flux density, flux tubes, Gauss' Law, etc. – however, the flux is now a *magnetic* flux instead of an *electric* flux, and it obeys slightly different properties.

Analysis of the magnetic field will lead us, eventually, to the creation of a new circuit element – the inductor. This is a device whose geometry, by design, fully exploits the properties of magnetic fields.

In future study we will see that the magnetic field is fundamental to modern civilisation. The magnetic field is a *strong* field compared to other forces of nature. It is used to generate power (turbine generators), it does useful work in industry (electric motors), commercial premises (air conditioning) the common household (refrigerators, vacuum cleaners, etc.), transportation (vehicles such as cars, trains, etc.) and, increasingly, robotics.

13.1 Background

The study of magnetic fields involves the study of phenomena which are in many ways similar to phenomena associated with the presence of electric fields. The concepts of action-at-a-distance, flux, flux density, field intensity etc. are important field-describing quantities in both cases. The nature and origins of the two types of field are, however, substantially different. Electrostatic fields have been seen to be associated with individual or groups of electric charges. The field is assumed to originate on positive charge and terminate on negative charge with the region in the neighbourhood of these charges becoming a region in which other introduced electric charges experience attractions or repulsions according to rules derived using Coulomb's Law and/or Gauss' Law.

Magnetic fields on the other hand originate whenever a charge experiences motion. Experiments show there to be a magnetic field surrounding any conductor (wire) carrying an electric current. (There must also be an electric field associated with these moving charges since moving charge implies that a force is causing the motion. This force is often referred to as the electromotive force, or “emf” for short). The magnetic field of a permanent magnet is the result of unpaired spin in some of the electrons of certain ferromagnetic materials – electron spin is charge motion. If a majority of the atoms of the ferromagnetic material can be made to align such that the direction of this unpaired spin is non-random, there will be a resultant magnetic field which will appear as permanent magnetism.

A fundamental law of magnetism, called Ampère's Law, will be used to derive expressions for the magnetic field as a function of position for several relatively simple and symmetrical arrangements of current-carrying conductors. An extension of Ampère's Law to cases involving wound magnetic cores will enable us to investigate the properties of a practical circuit device called the inductor.

We have seen that in electrostatics the cause of any electric field is the existence of an excess or deficiency of fundamental charge (electrons) in a particular region. Magnetic fields are produced wherever charge is in motion. The most common situation of charge motion that we shall encounter will be that involved in the movement of charge (current) along a conductor in response to the establishment of an electromotive force (voltage) across the conductor.

13.1.1 Experimental Results

An electric current produces a magnetic field

On 21 April 1820, the Danish scientist Ørsted noticed a compass needle deflected from magnetic north when an electric current from a battery was switched on and off, confirming a direct relationship between electricity and magnetism. Three months later he began more intensive investigations and soon thereafter published his findings[1], showing that an electric current produces a circular magnetic field.

The French experimentalist André-Marie Ampère began to work intensely on this new subject. He interpreted Ørsted's experiment and all magnetic phenomena already known for a long time as being due to an interaction between *current elements*[2]. To this end it was necessary to suppose the existence of electric currents inside the Earth and inside magnets. According to Ampère, these electric currents would be responsible for the so-called magnetic properties of these bodies. All these phenomena would be then due to a single principle, namely, the force between current-carrying conductors. With this new hypothesis, Ampère expected to explain and unify not only the magnetic phenomena known for a long time as the interaction between two magnets or the interaction between the Earth and a magnetic needle, but also the phenomenon discovered by Ørsted of a torque produced by a current-carrying wire and acting upon a magnetic needle. Moreover, from this hypothesis Ampère was able to

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[1] He wrote it in Latin, over four pages, sending it as a brochure to several scientists on 21 July 1820. It caused a sensation, being translated and published in several scientific journals.

[2] A *current element* is defined as a current over an infinitesimally small directed distance, $I\mathbf{dl}$.

predict a new phenomenon, not yet observed by anyone before him. This new phenomenon was the interaction between two current-carrying wires. He soon performed experiments showing the existence of this new interaction.

In 1822 Ampère arrived at his final mathematical expression describing the interaction between two current-carrying elements. With this expression he could explain the magnetic phenomena, Ørsted's discovery and all of his own experiments describing the torque and force which he observed between current-carrying wires. In November 1826 he published his main work on this subject: *Theory of Electrodynamic Phenomena, Uniquely Deduced from Experience*.

Interestingly, Ampère was the first person to conclude that current always exists in a closed loop – he observed the deflection of a magnetized needle when placed next to a battery, and was able to deduce the sense of the current inside the battery by the direction of the deflection of the magnetic needle.

13.1.2 Theoretical Results

It was the great James Clerk Maxwell that eventually reconciled electric and magnetic phenomena into one entity called electromagnetism in 1855. In doing so, he clearly had in mind a picture of force fields and lines of force as espoused by Michael Faraday. The mathematics relies on advanced *vector calculus*. Nevertheless, for simple geometries and static charges or steady currents, the equations of electromagnetism reduce to fairly simple practical formula that we can apply in everyday situations.

Since a current is charge in motion, it should come as no surprise that when Einstein's special theory of relativity came along in 1904, it was realised that Coulomb's Law, if modified to include charges in motion, will give terms which can be identified with a magnetic field. Since motion is relative, a given physical experiment which is purely electrostatic in one coordinate system can appear as electromagnetic in another coordinate system that is moving with respect to the first. Magnetic fields seem to appear and vanish merely by a change in the motion of the observer. Hence the subject of relativity plays a fundamental role in electromagnetics. It can be shown that all the laws of electromagnetic fields can be derived by applying the relativistic transformation to Coulomb's Law.

13.2 The Magnetic Field

Experimental results (such as detecting the magnetic field with a compass) show us that the magnetic field around a long straight conductor is *circular*, i.e. "lines of force", such as those exposed by iron filings sprinkled on a perpendicular surface to the wire, form circles. The magnetic field intensity is denoted by **H**. The direction of the magnetic field is the direction in which the north pole of a compass points. For a long straight conductor, the direction of the field is suitably described by the "right hand grip rule":

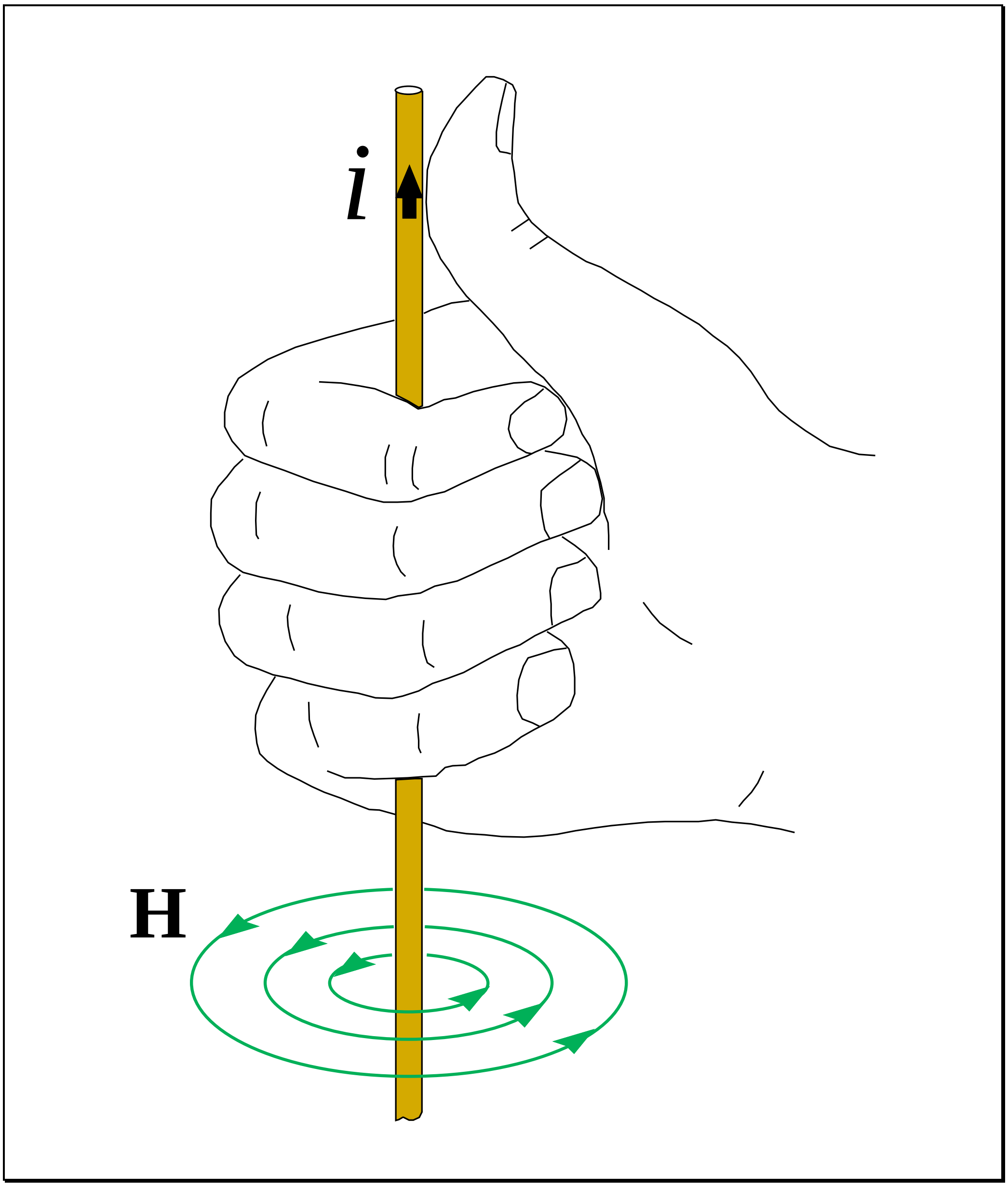


Figure 13.1 – The Right Hand Grip Rule for Magnetic Field Direction

13.3 Ampère's Law

In classical electromagnetism, Ampère's Law relates magnetic fields to electric currents that produce them:

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enclosed}} \tag{13.1}$$

Ampère's Law

The great Scottish physicist James Clerk Maxwell (not Ampère) derived it in his 1855 paper *On Faraday's Lines of Force*, based on an analogy to hydrodynamics.

The integration is a line integral around a closed path, l, which "encloses" the current I. To understand what this means, it is best to illustrate with examples.

13.3.1 The Magnetic Field Around an Infinitely Long Wire

Consider the magnetic field produced by current I in an infinitely long wire.

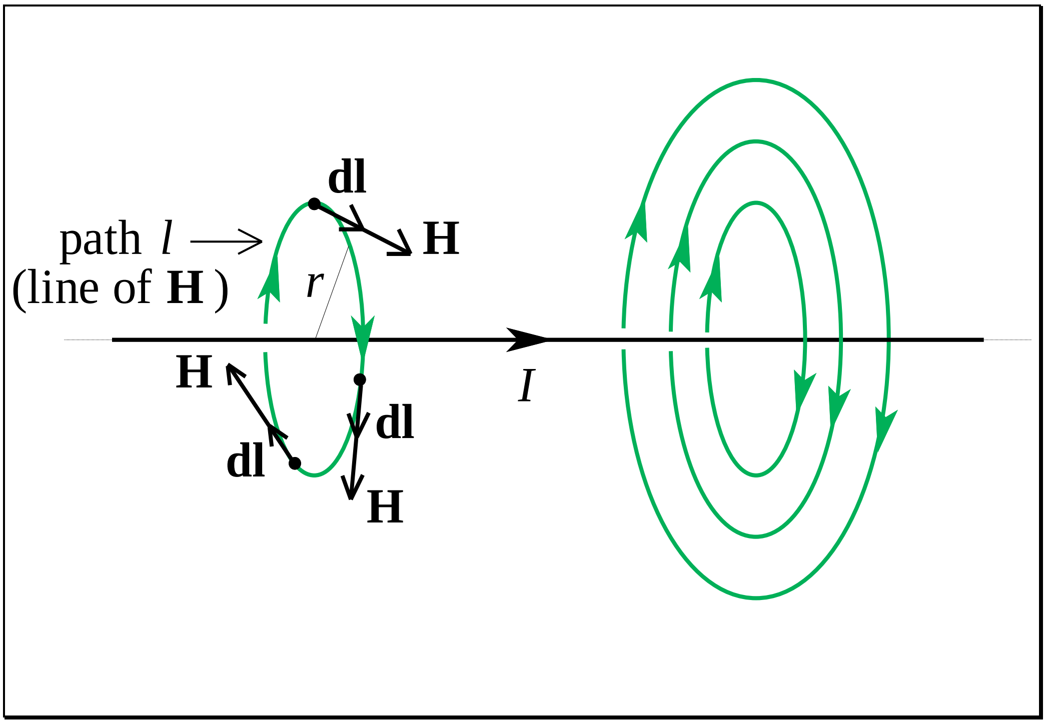


Figure 13.2

As shown in the figure (and which is experimentally verified), the **H** field forms concentric circles about the current-carrying wire. In applying Ampère's Law to this situation, we will choose an *arbitrary* path l which happens to coincide with a magnetic field line **H** at a radius *r* from the wire, and "integrate around the loop". We choose this arbitrary path to make the integration easier – at all points around the circular path, the direction of the magnetic field is tangential to the circumference of the circle, as is the differential path length vector, **dl**. Since the **H** vector and the **dl** vector always point in the same direction at any point on the path *l*, the dot product $\mathbf{H} \cdot \mathbf{dl}$ will reduce to Hdl. Ampère's Law then becomes:

$$\oint Hdl = I \tag{13.2}$$

where the current *I* is the current in the infinitely long wire, which has been "enclosed" by our path *l*. Now we again invoke symmetry arguments (similar to Gauss' Law) – there is no special orientation of a wire, or of a circle around the wire, and so we would expect the *magnitude* of the magnetic field to be constant at any point around a circular path *l* of fixed radius *r*. Then *H* is a constant in the integral, and it can be "brought out the front":

$$H \oint dl = I \tag{13.3}$$

Now the integral of *dl* is just *l*, the length of our path around the circle. Therefore:

$$Hl = H 2\pi r = I \tag{13.4}$$

We can now express the magnitude of the magnetic field as:

$$H = \frac{I}{2\pi r} \tag{13.5}$$

Finally, defining the unit vector $\hat{\mathbf{h}}$ as always pointing tangentially to the circumference of a circle of radius *r* centred on the wire, in the direction of the right-hand grip rule, we have a formula for the magnetic field around an infinite current-carrying conductor:

$$\mathbf{H} = \frac{I}{2\pi r}\hat{\mathbf{h}} \quad \text{Am}^{-1} \tag{13.6}$$

13.3.2 The Contour Integral

Integration around the path *l* is known as a *contour integral*. In more advanced electromagnetic analysis, which invokes the branch of mathematics known as *vector calculus*, the contour integral is shown to be related to a *surface integral*. At this introductory stage of electromagnetism, it is worthwhile just to take a glimpse at the concept.

You can think of the path *l* as a child's soap bubble blower, and imagine a thin sheet of detergent across the circle. The current (in the wire) then pierces the thin sheet of detergent. In fact, you can distort the soap bubble into any shape you want – the current will still pierce the soap bubble and it is only the periphery of the soap bubble (the sturdy plastic ring) which is important in performing Ampère's Law:

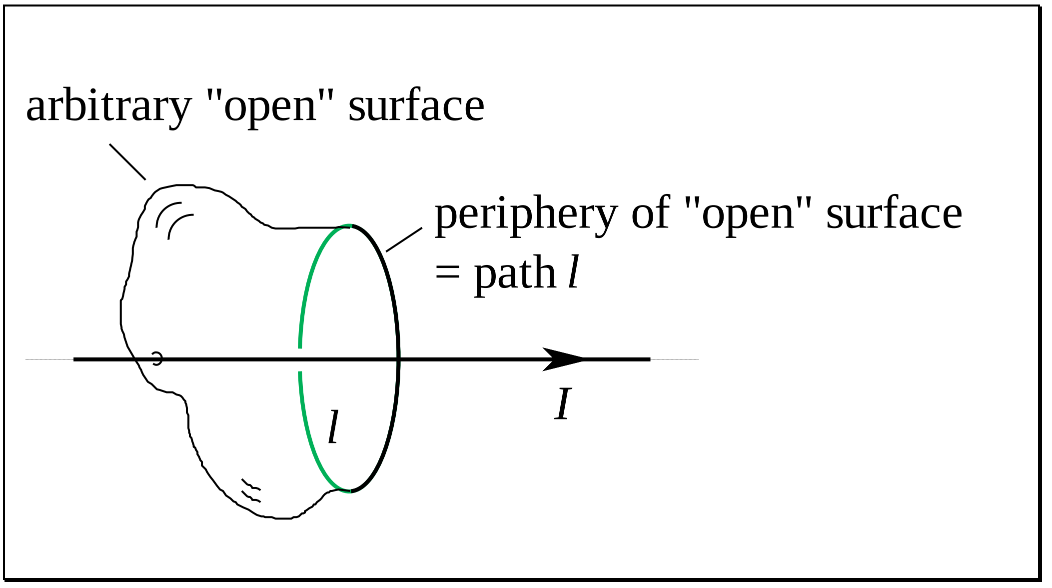


Figure 13.3

So we can imagine "enclosing the current by path *l*" as "choosing an open surface so that current comes out of the open part of the surface". The periphery of the open surface then forms the closed path in Ampère's Law. In other words, once current pierces and enters the soap bubble, it has to exit through the sturdy plastic ring. This kind of thinking will help later with more intricate geometries, where you may be need to imagine a current "piercing" the soap bubble multiple times, and therefore contributing to the right-hand side of Ampère's Law more than once.

13.3.3 Arbitrary Paths

Ampère's Law for the infinitely long wire is valid not only for a simple circular path which coincides with the circular **H** field of the infinite wire but also for arbitrarily shaped closed loops about the current *I*. To demonstrate this, consider the figure below which shows a current *I* inside a noncircular closed path:

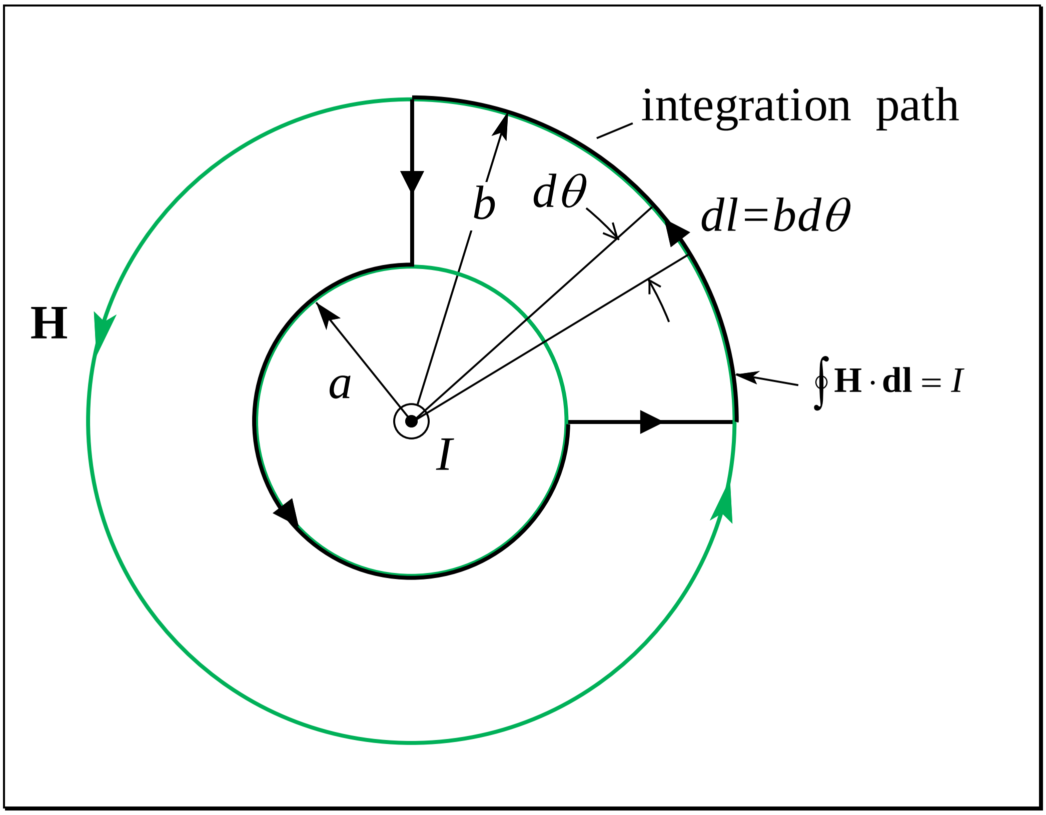


Figure 13.4

In this figure, the “dot” in the centre of the wire represents current “out of the page”, and the wire is perpendicular to the surface of the page. Integrating the **H** field around the path, we obtain:

$$\begin{aligned} \oint \mathbf{H}\cdot d\mathbf{l} &= \int_0^{\pi/2} \frac{I}{2\pi b}(b\,d\theta) + \int_{\pi/2}^{2\pi} \frac{I}{2\pi a}(a\,d\theta) \\ &= \frac{I}{2\pi}\frac{\pi}{2} + \frac{I}{2\pi}\frac{3\pi}{2} \\ &= I \end{aligned} \tag{13.7}$$

Note that the radial segments of the path do not contribute to the integral because there the **H** field is normal (perpendicular) to the path and the dot product is zero. Hence the line integral of **H** around any closed path is equal to *I*, because a closed path of arbitrary shape can always be replaced by many small arcs and radial segments.

What if the integration path does not enclose current, as for example, the path shown below?

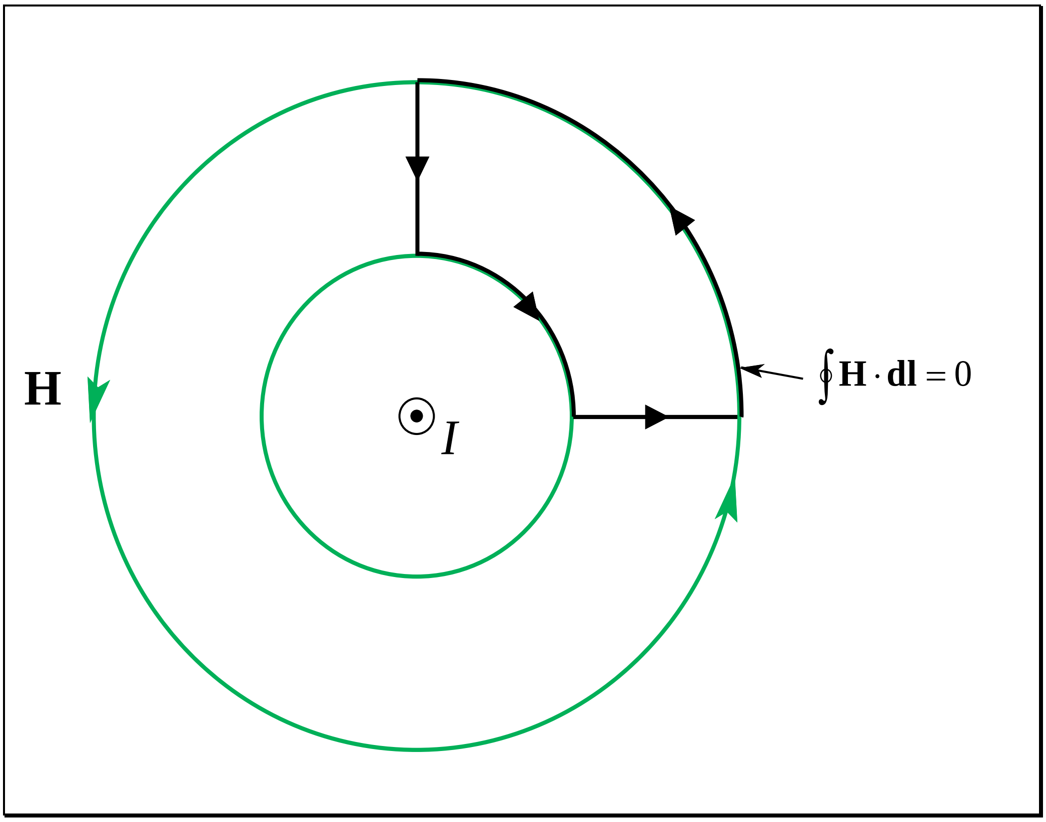


Figure 13.5

The right-hand side of Ampère's Law is then zero, because the contribution to the integral along the outer arc cancels that of the inner arc. That is, the outer arc gives a value of $I/4$, whereas the inner one yields $-I/4$. The signs are different because along the outer arc the **H** field is along the direction of integration, whereas along the inner arc **H** and **dl** are opposite in direction.

Ampère's Law is valid for any magnetic field and *steady* current (not just for fields produced by a current in an infinitely long wire). (13.8)

13.3.1 The Magnetic Field Around a Toroid

A toroidal coil

Consider a toroidal coil of *N* turns carrying a current *I*:

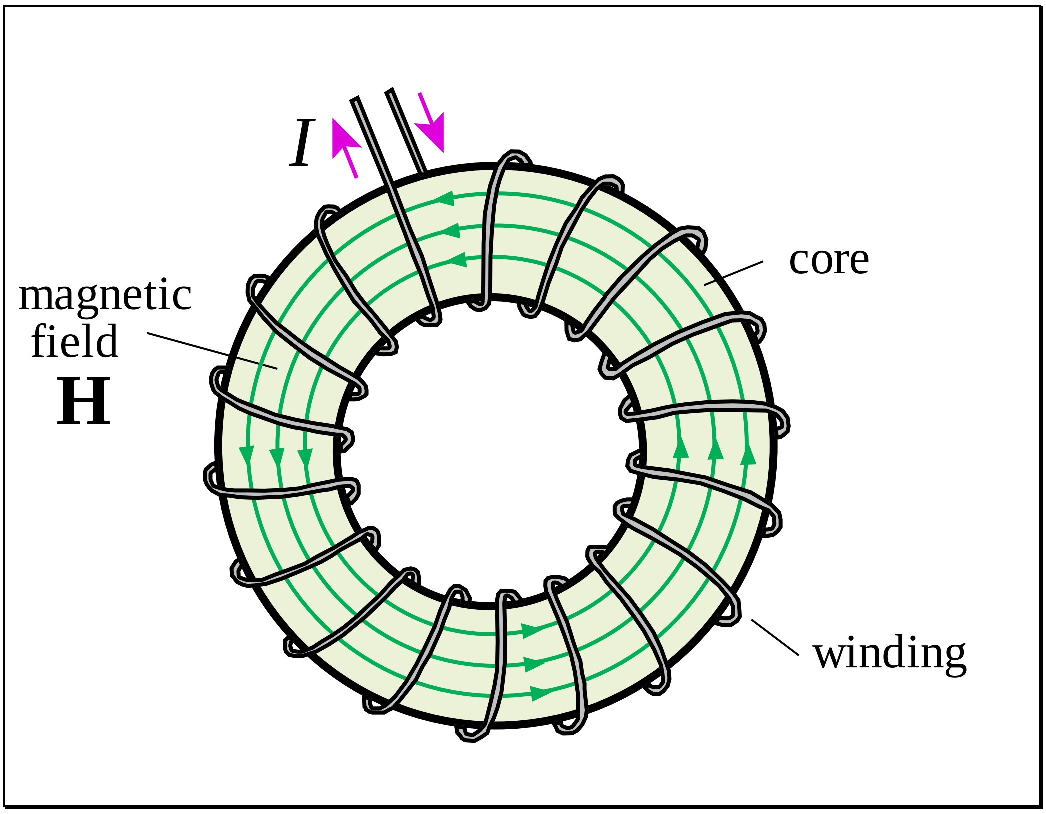


Figure 13.6

The geometry of this example is shown below:

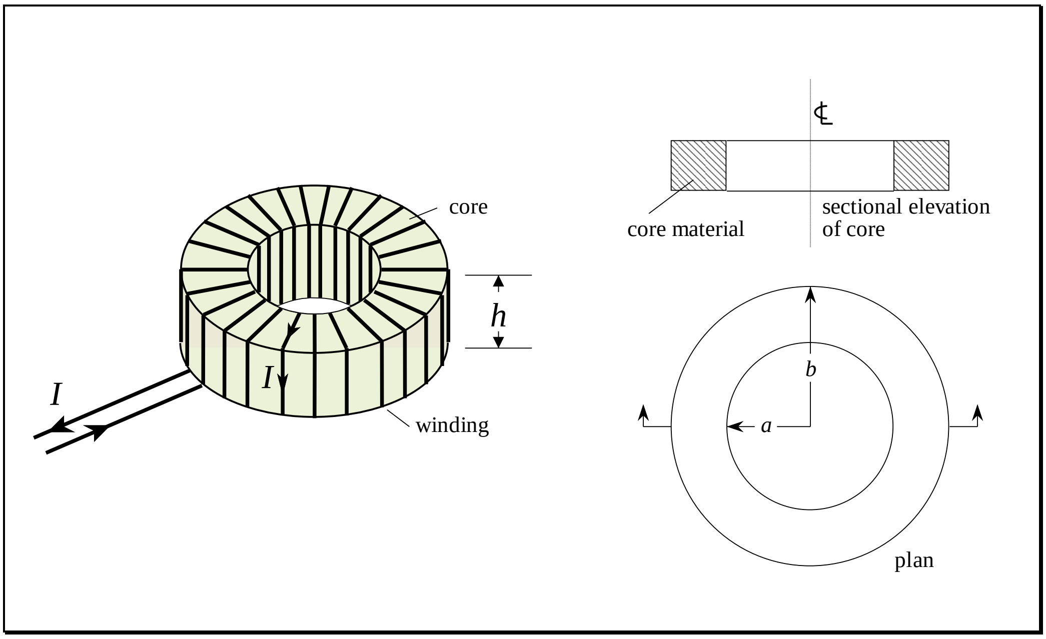


Figure 13.7

In the figure below, we show a cross section of the toroid with current I in the N turns of the winding:

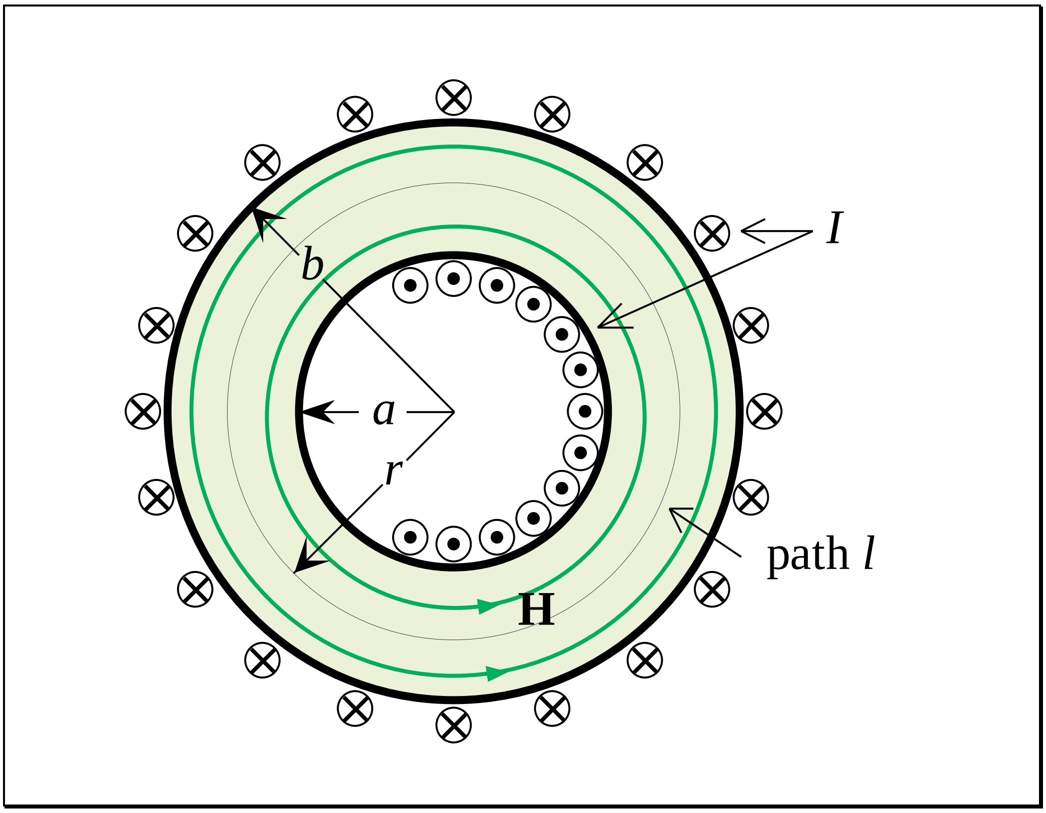


Figure 13.8

To analyse this arrangement using simple mathematics, we have to assume that the winding is distributed uniformly around the toroid, and there are no gaps (unlike in the figure) – i.e. the loops of wire are infinitesimally close together with the effect that the windings form a uniform "current sheet" on the interior and exterior of the core. If we make this assumption, then the **H** field is uniform within the interior of the core (if we did not make this assumption, the field would be "bumpy" due to the windings). In practice, this assumption gives reasonably accurate values for the magnetic field (less than 10% error for tight windings).

We can now evaluate the left-hand side of Ampère's Law around a closed path l at a radius r from the centre of the toroid:

$$\oint \mathbf{H} \cdot d\mathbf{l} = \oint H dl = H \oint dl = Hl = H 2\pi r \tag{13.9}$$

13.14

The right-hand side of Ampère's Law is the current "enclosed" by the closed path. In this case, we "enclose" a current equal to N times the current in an individual winding, since there are N turns. We therefore have:

$$H 2\pi r = NI \tag{13.10}$$

and so the magnetic field intensity in the interior of a toroid is:

$$\boxed{\mathbf{H} = \frac{NI}{2\pi r}\hat{\mathbf{h}} \quad \mathrm{a} < \mathrm{r} < \mathrm{b}} \tag{13.11}$$

This is the same as the infinite wire, just increased N times.

If we choose a circular path with $r \le a$ then $\oint \mathbf{H} \cdot \mathbf{dl} = 0$ because no current is enclosed.

If we choose a circular path with $r \ge b$ then Ampère's Law tells us that:

$$\begin{aligned} \oint \mathbf{H} \cdot \mathbf{dl} &= N(I - I) = 0 \\ H 2\pi r &= 0 \\ H &= 0 \end{aligned} \tag{13.12}$$

that is, the field outside a toroid is zero (ideally) because no net current is enclosed.

Thus, Ampère's Law tells us that the **H** field is confined to the interior of the toroid where it varies inversely with distance *r* from the centre:

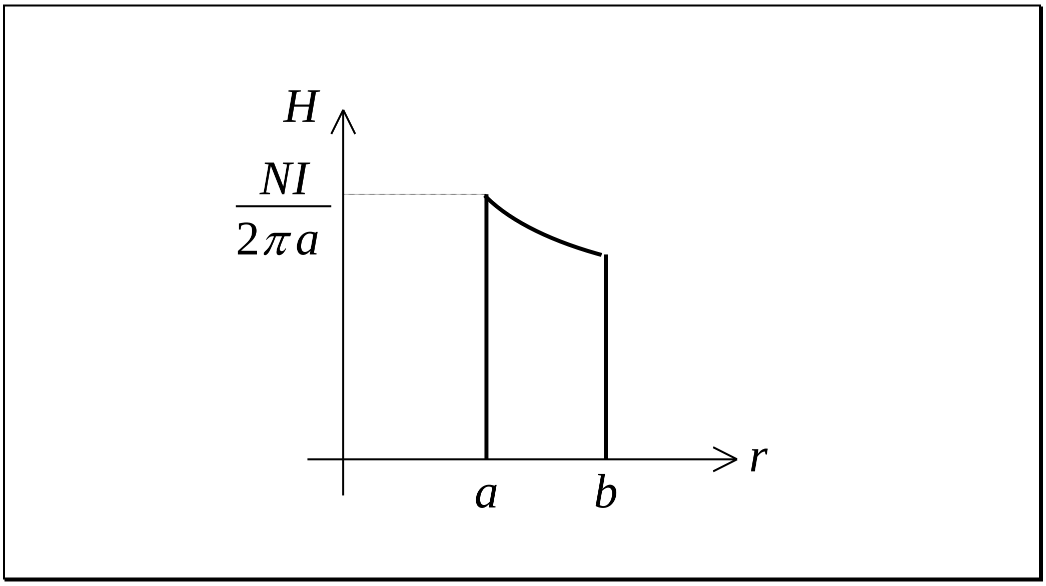


Figure 13.9

13.3.2 The Magnetic Field of a Solenoid

A configuration used to produce strong magnetic fields is a helical coil called a *solenoid*:

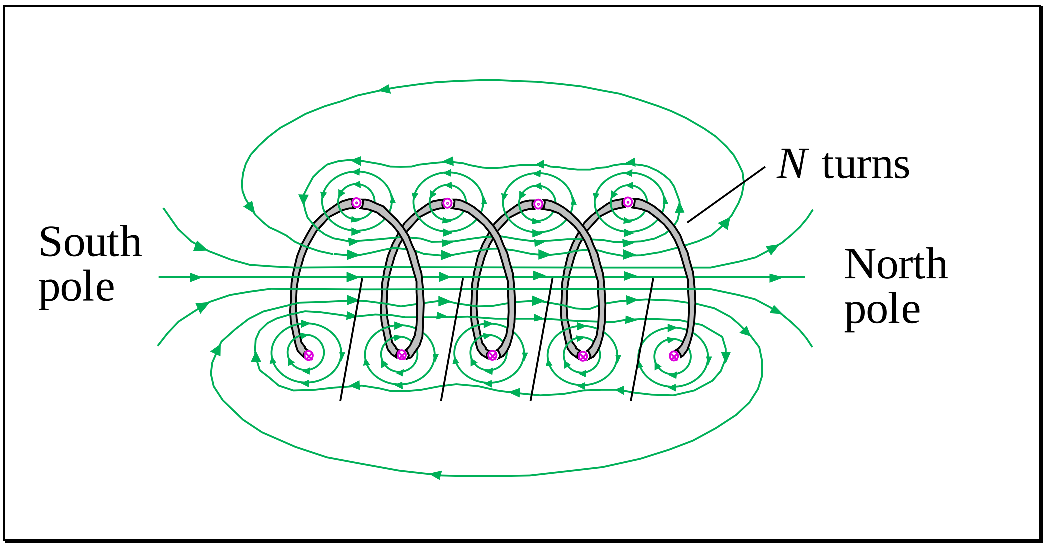


Figure 13.10

The solenoid is useful because the magnetic field "comes out the ends" and is basically an electromagnet with a N and S pole.

To analyse the solenoid using Ampère's Law, we assume that the solenoid is a tightly wound helix of conducting wire – so tight that the individual turns are infinitesimally close to one another, and we have in effect a “current sheet” that circulates around the diameter of the solenoid (no bumps in the field!). We also assume an infinite length so there are no “end effects”.

The analysis will reveal the magnetic field inside the solenoid:

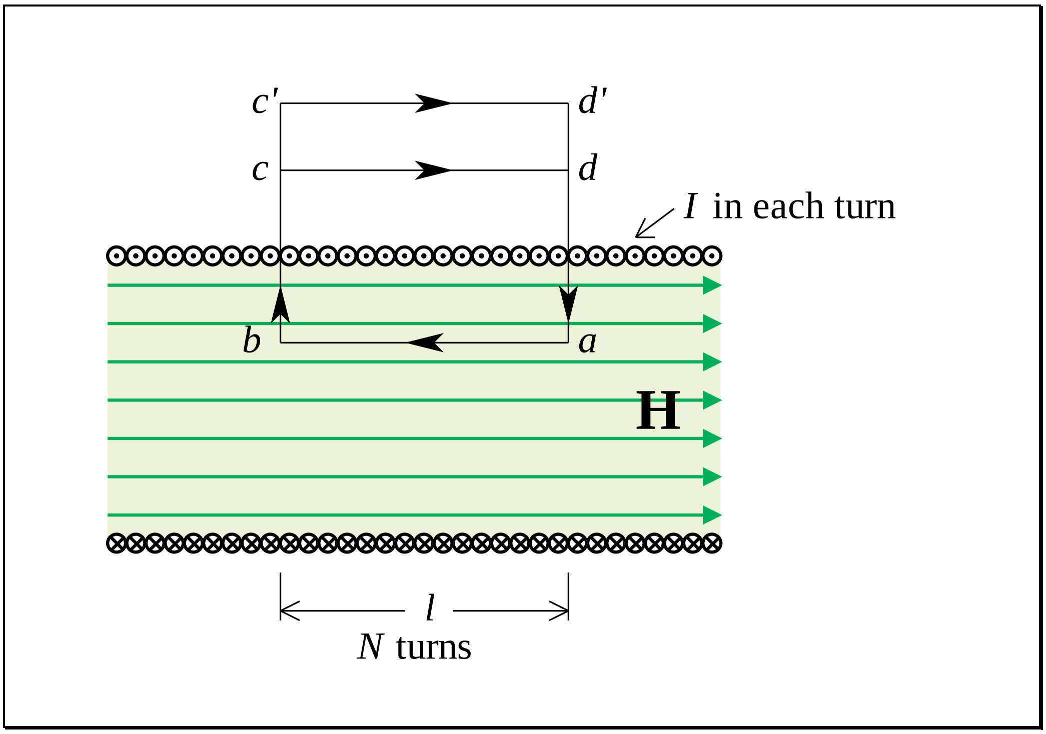


Figure 13.11

For an infinite solenoid the magnetic field everywhere inside must be parallel to the axis of the solenoid – otherwise the magnetic field would “come out” somewhere, and the solenoid is not infinite in length. We also claim that the magnetic field outside the solenoid is zero; therefore along path *cd* the line integral $\int_c^d \mathbf{H} \cdot d\mathbf{l}$ is zero. If it is not, and a finite value along *cd* is obtained, then this would imply that the same value along *c'd'* must be obtained. This in turn implies that a constant field exists everywhere outside the coil which is contrary to experimental observation.

Ampère's Law gives us:

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_a^b \mathbf{H} \cdot d\mathbf{l} + \int_b^c \mathbf{H} \cdot d\mathbf{l} + \int_c^d \mathbf{H} \cdot d\mathbf{l} + \int_d^a \mathbf{H} \cdot d\mathbf{l} \\ = NI \tag{13.13}$$

Since the field along *cd* is zero, and along *bc* and *da* the **H** field is at right angles to the path, we are left with:

$$\int_a^b \mathbf{H} \cdot d\mathbf{l} = NI \\ H = \frac{NI}{l} \tag{13.14}$$

This infinite solenoid result is an excellent approximation for the field of long solenoids. It shows that **H** is independent of the solenoid diameter, and that **H** is constant over the cross section of the solenoid.

It is easy to show that the magnetic field intensity at either end of a long solenoid must be half that at its centre. To see this, cut an infinite solenoid in two. If the current in the two parts is maintained at the same value, the **H** field at the newly created ends must drop to one-half its original value, otherwise the **H** field would not have its original value when the two ends are reconnected. This implies that half the field lines that exist at the centre of a long solenoid leak out through the solenoid turns somewhere between the centre and one end, as shown in Figure 13.10.

13.4 Magnetic Flux and Flux Density

Magnetic flux postulated

To explain magnetic phenomena, 19th century scientists invoked an analogy with fluids and postulated the existence of a magnetic fluid, known as magnetic "flux", Φ, which streamed throughout space and manifested itself as magnetism. Magnetic flux always streams out of north poles and into south poles, and forms a closed loop. We still use this concept of flux today, as the theory has been spectacularly successful.

Drawing on all the ideas used to explain electrostatics, we can now state:

13.4.1 Flux Tubes

Tubes of magnetic flux Φ stream through space using the **H** field lines as boundaries.

13.4.2 Flux Density

There must be a magnetic field that measures the density of the magnetic flux at each point. We define the magnetic flux density as the vector **B**. It's relationship to **H** is given by:

Magnetic flux density is related to magnetic field intensity

$$\mathbf{B} = \mu\mathbf{H} \quad \mathrm{T} \tag{13.15}$$

The units of magnetic flux density are the tesla, abbreviated T.

13.4.3 Permeability

The magnetic constant of the medium is the permeability:

Permeability as a magnetic property of the medium

$$\mu = \text{permeability of the medium} \tag{13.16}$$

$$\mu_0 = \text{permeability of free sapce} = 4\pi \times 10^{-7}\ \mathrm{Hm}^{-1} \tag{13.17}$$

We also define the *relative permeability* of a material:

Relative permeability defined

$$\mu_r = \mu / \mu_0 \tag{13.18}$$

Note that ferromagnetic materials have relative permeabilities in the thousands.

13.4.4 Flux

The magnetic flux streaming through an area is given by:

$$\Phi = \int \mathbf{B} \cdot \mathbf{dA} \quad \text{Wb} \tag{13.19}$$

The units of magnetic flux are the weber, abbreviated Wb.

13.4.5 Gauss' Law for Magnetism

There are no sources or sinks of magnetic flux (no "magnetic monopoles") since lines of **H** form closed paths. Gauss' Law for magnetism is then:

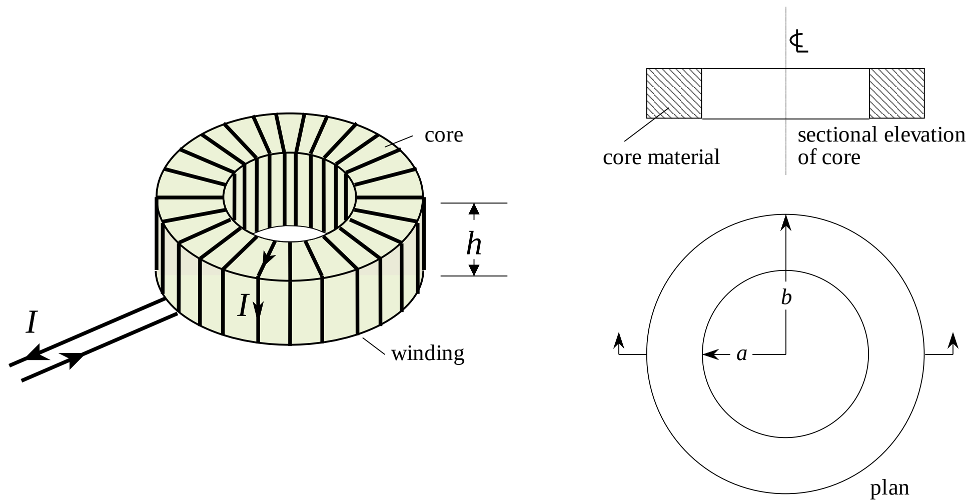
Gauss' Law for magnetism

$$\oint \mathbf{B} \cdot \mathbf{dA} = 0 \tag{13.20}$$

where you should note that the integral is performed on a closed surface. This means that if flux enters your closed surface, it must exit.

EXAMPLE 13.1 Magnetic Flux Density and Flux in a Toroid

We return to the toroid:



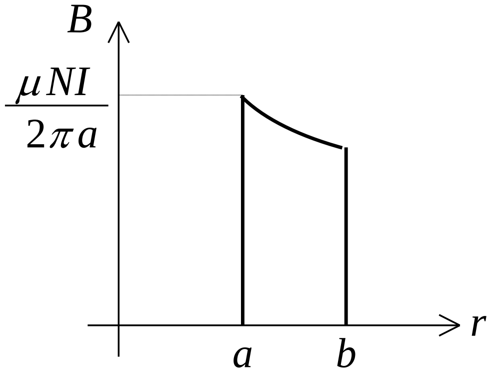
for which we found:

$$\mathbf{H} = \frac{NI}{2\pi r}\hat{\mathbf{h}} \quad a < r < b$$

The magnetic flux density **B** within the toroidal coil will depend upon the material of the core. If the core is constructed from a magnetic material with a relative permeability μ_r then the flux density within the core is given by:

$$\mathbf{B} = \mu\mathbf{H} = \frac{\mu_r \mu_0 NI}{2\pi r}\hat{\mathbf{h}} \quad a < r < b$$

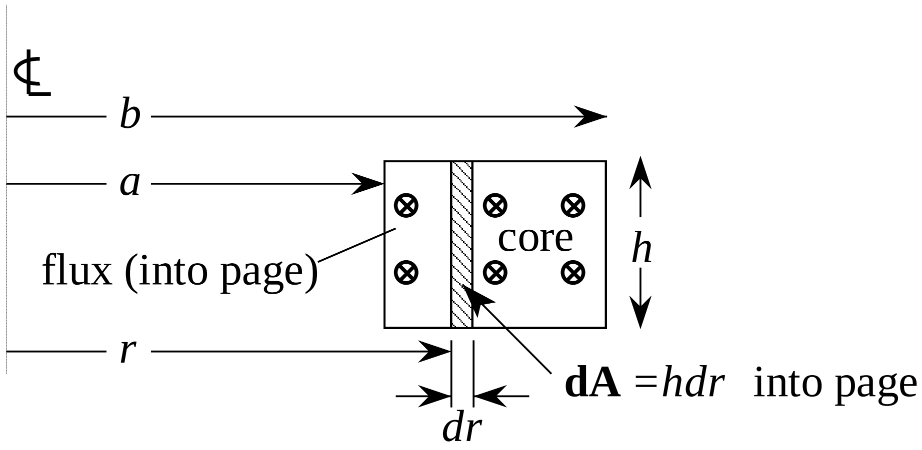
The flux density **B** therefore varies in the same manner as **H** with distance *r* from the centre of the toroid:



The magnetic flux streaming around the interior of the toroid is given by:

$$\Phi = \int \mathbf{B} \cdot \mathbf{dA}$$

where we have to setup up an area A to intercept the entire flux stream. To evaluate this, consider the right-half cross-section of the core:



In this picture, the flux is directed into the page, as is the differential surface area **dA** used to "capture" the flux. The magnitude of the differential area is $dA = hdr$. The flux integral is now:

$$\begin{aligned}\Phi &= \int_a^b Bhdr \\ &= \int_a^b \frac{\mu_r \mu_0 NI}{2\pi r} hdr \\ &= \frac{\mu_r \mu_0 NIh}{2\pi} \int_a^b \frac{dr}{r} \\ &= \frac{\mu_r \mu_0 NIh}{2\pi} \ln\left(\frac{b}{a}\right) \quad \text{Wb}\end{aligned}$$

13.22

An approximate approach to this exact method for the determination of **B** and Φ will now be considered. The approximation hinges on the assumption that the flux density does not vary too much between the inner and outer radii of the toroid, so it can be assumed to be constant and equal to the flux density at the mid-radius:

$$r_{\text{mid}} = \frac{a+b}{2}$$

Thus:

$$\begin{aligned}\mathbf{B}_{\text{mid}} &= \frac{\mu_r \mu_0 NI}{2\pi r_{\text{mid}}}\hat{\mathbf{h}} \\ &= \frac{\mu_r \mu_0 NI}{\pi(a+b)}\hat{\mathbf{h}} \quad \text{T}\end{aligned}$$

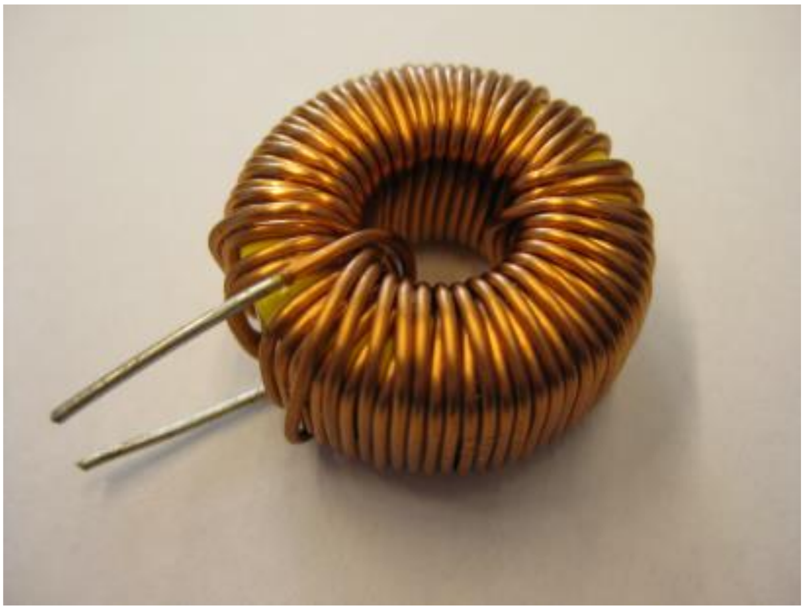
With the approximation that the flux density is uniform across the core cross-section, the flux is given simply by:

$$\begin{aligned}\Phi &= \int \mathbf{B} \cdot \mathbf{dA} \approx B_{\text{mid}} A_{\text{core}} = B_{\text{mid}} h(b-a) \\ &= \frac{\mu_r \mu_0 NIh}{\pi}\left(\frac{b-a}{b+a}\right) \quad \text{Wb}\end{aligned}$$

The proportional error resulting from this approach is found by subtracting one result from the other and dividing by the true result. You will observe that the factor $\frac{\mu_r \mu_0 NIh}{2\pi}$ will cancel when this procedure is followed. Thus the proportional error becomes:

$$\varepsilon = \frac{\left(\frac{b-a}{b+a}\right) - \frac{1}{2}\ln\left(\frac{b}{a}\right)}{\frac{1}{2}\ln\left(\frac{b}{a}\right)}$$

For a practical toroid the difference between the radii a and b will not be large. For the toroidal coil shown below:



the dimensions are:

$$a = 6 \text{ mm}$$
$$b = 14 \text{ mm}$$

which gives a proportional error of:

$$\varepsilon = \frac{\left(\frac{2}{5}\right) - \frac{1}{2}\ln\left(\frac{7}{3}\right)}{\frac{1}{2}\ln\left(\frac{7}{3}\right)} = \frac{0.4 - 0.4236}{0.4236} = -0.05582 \approx -6\%$$

This gives us justification for the practical approach given in the next section.

13.5 Magnetic Circuits

Consider an arrangement where a coil of wire is wound onto an iron core that has a small air gap cut into it:

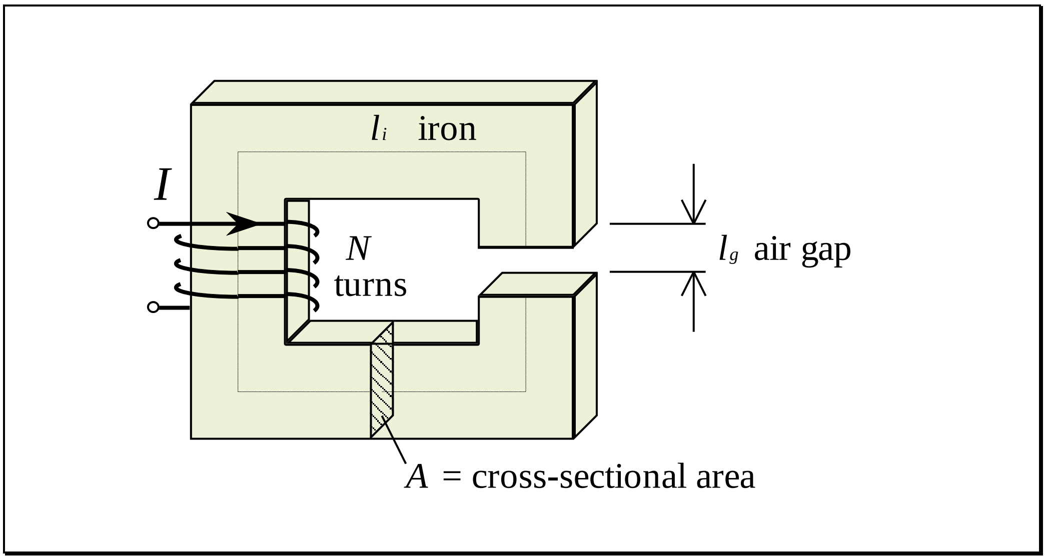


Figure 13.12

We will make the following assumptions:

1. Magnetic flux distributes itself evenly across the cross section of any magnetic core material through which it passes.
2. The *mean path length* of a flux path is assumed to be the median value.
3. Magnetic flux will follow the path of maximum relative permeability.
4. There is no "fringing" of the field at air gaps, i.e. the flux density of an air gap is the same as at the "poles" of the iron at either side of the air gap.

An illustration of a "fringing field" due to an air gap is shown below:

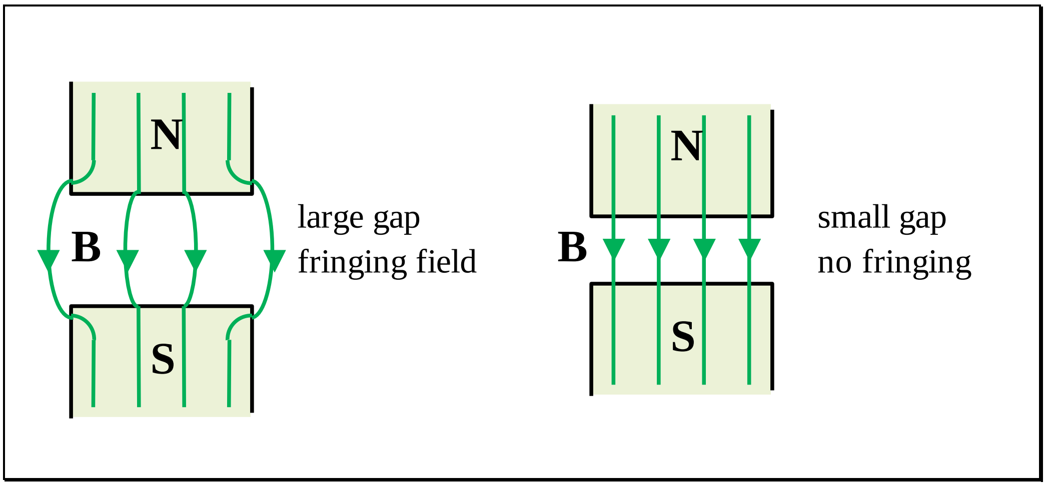


Figure 13.13

Using these 4 assumptions, we can now write Ampère's Law by following the mean path length l around the magnetic circuit:

$$\begin{aligned} NI &= \oint \mathbf{H} \cdot d\mathbf{l} \\ &= \int_{l_i} H_i dl_i + \int_{l_g} H_g dl_g \\ &= H_i l_i + H_g l_g \end{aligned} \tag{13.21}$$

The left hand-side just reflects the fact that the path of integration enclosed the N turns of the coil which carries current I. The right-hand side is just an expansion of $\oint \mathbf{H} \cdot d\mathbf{l}$ showing the individual contributions made by the path in iron and the path in air.

The dot product reduced to a scalar product since the infinitesimally small path lengths **dl** follow the magnetic field **H**, so they both have the same direction. This is a direct result of having a ferromagnetic material to direct the flux through a well-defined path.

The integral turned into a summation because we assume that **H** is uniform throughout the material. In all cases we will let the subscript i mean iron (or any ferromagnetic material) and subscript g mean gap (in air).

Substituting $H = B/\mu$ we get:

$$NI = \frac{B_i}{\mu_r \mu_0} l_i + \frac{B_g}{\mu_0} l_g \tag{13.22}$$

Note that the air gap has $\mu_r = 1$. We now make use of our assumption that there is no fringing at the air gap, in other words, $B_g = B_i = B = \Phi/A$:

$$NI = \frac{l_i}{\mu_r \mu_0 A} \Phi + \frac{l_g}{\mu_0 A} \Phi \tag{13.23}$$

As will be shown shortly, this looks like the magnetic analog of KVL, taken around a circuit consisting of a DC voltage source and two resistors. We will therefore exploit this analogy and develop the concept of *reluctance* and *mmf*.

If we define magnetic reluctance as:

Reluctance defined

$$\mathscr{R} = \frac{l}{\mu A} \tag{13.24}$$

and magnetomotive force (mmf) as:

Magnetomotive force (mmf) defined

$$\mathscr{F} = NI \tag{13.25}$$

then Ampère's Law, Eq. (13.23), gives:

Ampère's Law looks like a "magnetic Ohm's Law" for this simple case

$$\begin{aligned} \mathscr{F} &= \mathscr{R}_i \Phi + \mathscr{R}_g \Phi \\ &= \mathscr{R}_T \Phi \end{aligned} \tag{13.26}$$

This is analogous to Ohm's Law. It should be emphasised that this is only true where μ is a constant. That is, it only applies when the material is linear or assumed to be linear (**B** is proportional to **H**).

The magnetic analog to KVL is Ampère's Law. *What is the magnetic analog to KCL?* In simple systems where the flux path is known, all the magnetic flux streaming into some part of a structure must also stream out of another part of the structure. The analog to KCL for magnetic circuits is therefore Gauss' Law.

Gauss' Law is the magnetic analog of KCL

The two laws we will use for *magnetic circuits* are:

$$\sum_l \mathscr{F} = \sum_l \mathscr{R}\Phi, \quad \text{around a loop } l \tag{13.27}$$

$$\sum \Phi = 0, \quad \text{at a node} \tag{13.28}$$

Ampère's Law and Gauss' Law are simple summations for magnetic circuits

Compare this with KVL and KCL for *electric circuits*:

$$\sum_l V_s = \sum_l RI, \quad \text{around a loop } l \tag{13.29}$$

$$\sum I = 0, \quad \text{at a node} \tag{13.30}$$

KVL and KCL

In KVL above, V_s represents DC voltage sources (emfs), and RI are drops in potential due to resistive elements.

To extend the above analogy with KVL even further, we introduce the concept of *magnetic scalar potential*, denoted by U, which is the magnetic analog of an electric potential difference:

$$U = Hl = \mathscr{R}\Phi \tag{13.31}$$

Magnetic scalar potential defined

Compare this with Ohm's Law:

$$V = RI \tag{13.32}$$

Thus, the original magnetic arrangement can be redrawn in schematic notation as a magnetic circuit:

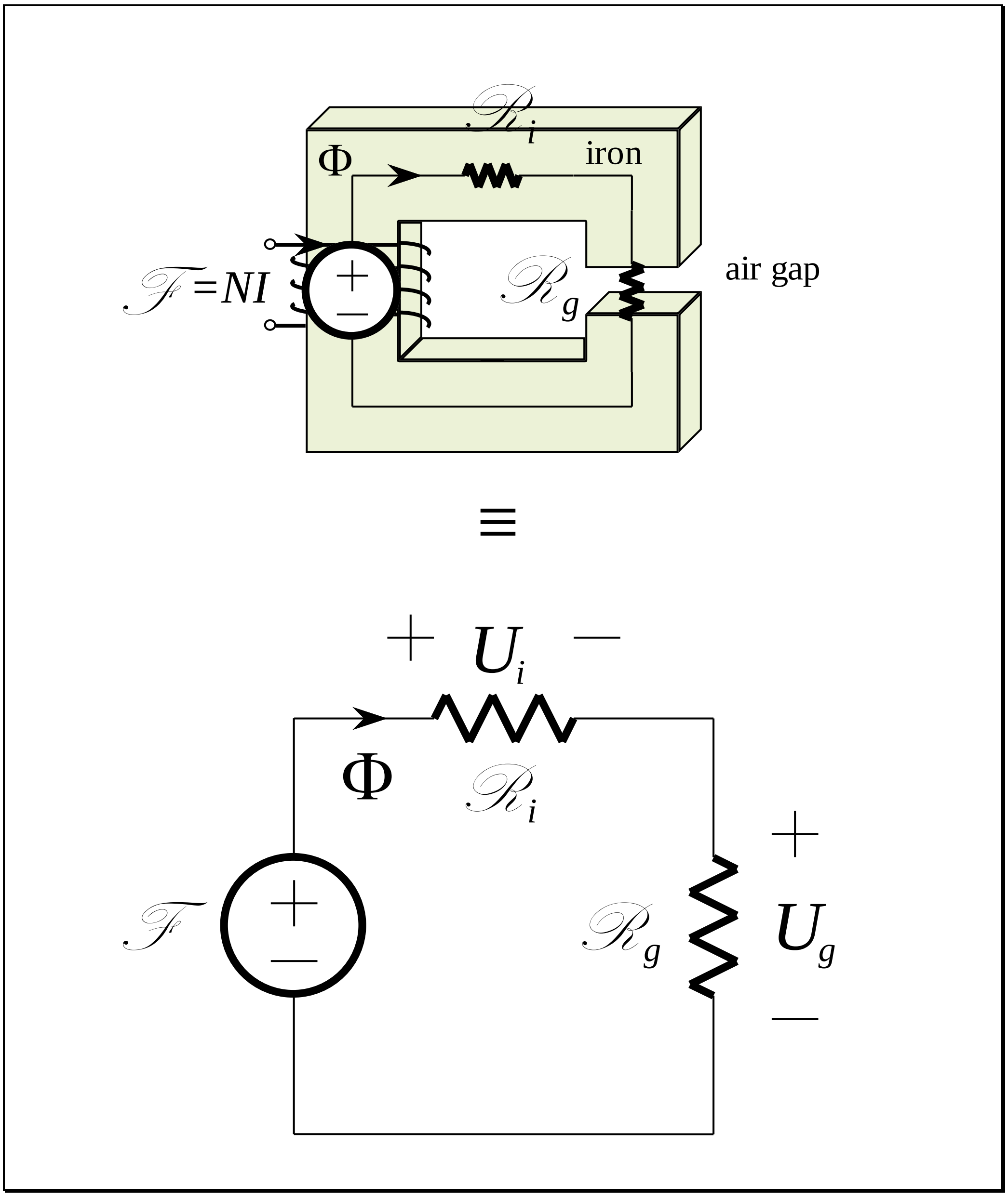
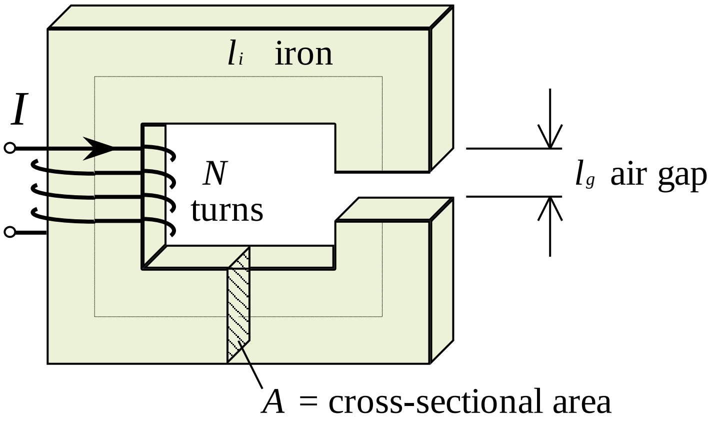


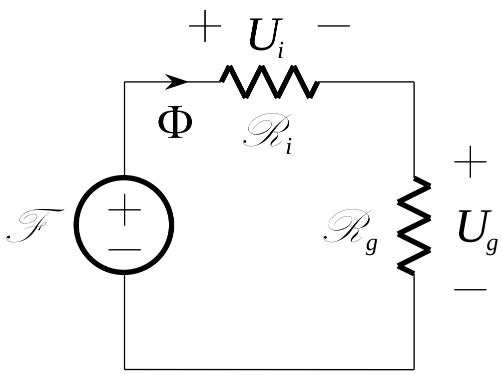
Figure 13.14

EXAMPLE 13.2 A Magnetic Circuit with an Air Gap

A magnetic circuit core, shown in the figure below, is constructed of commercial iron with a relative permeability of 3000. The cross section of the core is uniform and measures 40 mm x 40 mm. The mean path length in the iron is $l_i = 600\ \text{mm}$. The air gap has a length $l_g = 400\ \mu\text{m}$. The winding has 500 turns. We wish to compute the current required to produce a flux of 1.6 mWb.



We draw the equivalent magnetic circuit:



We write Ampère's Law around the magnetic circuit:

$$\begin{aligned}\mathcal{F} &= U_i + U_g \\ &= \mathcal{R}_i\Phi + \mathcal{R}_g\Phi \\ &= \mathcal{R}_T\,\Phi\end{aligned}$$

13.30

The reluctances are:

$$\mathcal{R}_i = \frac{l_i}{\mu_r \mu_0 A_i} = \frac{0.6}{3000 \times 4\pi \times 10^{-7} \times 0.04^2} = 99{,}472 \text{ AWb}^{-1}$$

$$\mathcal{R}_g = \frac{l_g}{\mu_0 A_g} = \frac{4 \times 10^{-4}}{4\pi \times 10^{-7} \times 0.04^2} = 198{,}944 \text{ AWb}^{-1}$$

The total reluctance is:

$$\mathcal{R}_T = \mathcal{R}_i + \mathcal{R}_g = 298{,}416 \text{ AWb}^{-1}$$

The mmf is then:

$$\mathcal{F} = NI = \mathcal{R}_T \Phi = 298416 \times 1.6 \times 10^{-3} = 477.5 \text{ A}$$

Therefore the current required is:

$$I = \frac{\mathcal{F}}{N} = \frac{477.5}{500} = 954.9 \text{ mA}$$

13.6 Summary

- The magnetic field intensity is denoted by **H**. Magnetic field lines form closed loops.

- The direction of the magnetic field around a current-carrying wire is determined by the right hand grip rule.

- Ampère's Law is:

$$\oint \mathbf{H} \cdot \mathbf{dl} = I_{\text{enclosed}}$$

- The magnetic field intensity at a perpendicular distance r from an infinitely long current-carrying conductor is:

$$\mathbf{H} = \frac{I}{2\pi r}\hat{\mathbf{h}}$$

- The magnetic field intensity in the core of a toroid at a distance r from the axis with N turns carrying current I is:

$$\mathbf{H} = \frac{NI}{2\pi r}\hat{\mathbf{h}}$$

- The fundamental relationship between magnetic flux density and magnetic field intensity is:

$$\mathbf{B} = \mu \mathbf{H}$$

  where μ is the permeability of the material.

- The magnetic flux streaming through an area is given by:

$$\Phi = \int \mathbf{B} \cdot \mathbf{dA}$$

- Gauss' Law for magnetism is:

$$\oint \mathbf{B} \cdot \mathbf{dA} = 0$$

- The magnetic reluctance of a structure is:

$$\mathcal{R} = \frac{l}{\mu A}$$

- The magnetomotive force (mmf) is defined as:

$$\mathcal{F} = NI$$

- The magnetic scalar potential is defined as:

$$U = Hl = \mathcal{R}\Phi$$

- A magnetic circuit can be schematically represented in a manner similar to an electric circuit.

- Analogous electric and magnetic circuit quantities are:

Electric	**Magnetic**
Electric field intensity $\mathbf{E}$	Magnetic field intensity $\mathbf{H}$
Current density $\mathbf{J} = \sigma\mathbf{E}$	Flux density $\mathbf{B} = \mu\mathbf{H}$
Conductivity σ	Permeability μ
Current I	Flux Φ
emf $V = \oint \mathbf{E} \cdot d\mathbf{l} \approx El$	mmf $\mathcal{F} = \oint \mathbf{H} \cdot d\mathbf{l} \approx Hl$
Ohm's Law $V = RI$	$U = \mathcal{R}\Phi$
Resistance $R = \frac{l}{\sigma A}$	Reluctance $\mathcal{R} = \frac{l}{\mu A}$
Kirchhoff's Voltage Law	Ampère's Law
Kirchhoff's Current Law	Gauss' Law

13.7 References

Beard, G.: *Electrostatics and Magnetostatics*, NSWIT, 1985.

Plonus, M.: *Applied Electromagnetics*, McGraw-Hill, 1984.

Exercises

1.

Calculate the magnetic field intensity **H** and magnetic flux density **B** at a distance of 1 m from a wire which carries a current of 1 A if:

(a) The wire is immersed in free space.

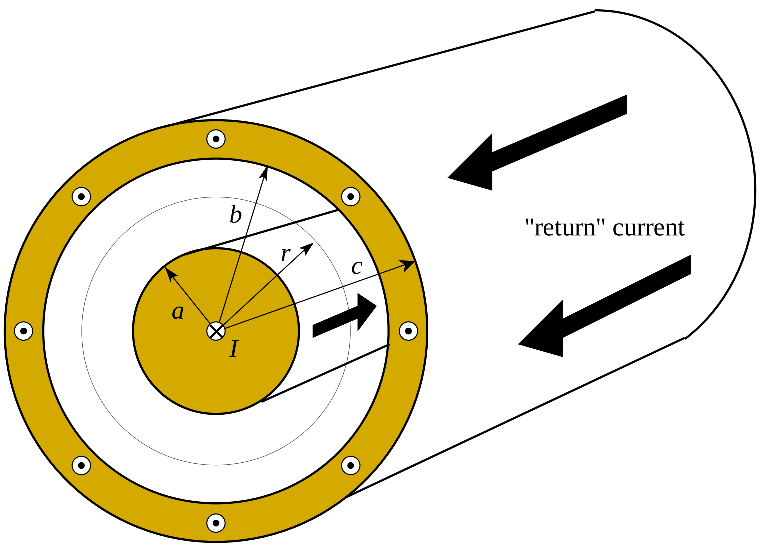
(b) The wire is immersed in a magnetic material with relative permeability $\mu_r = 100$.

2.

Determine the magnetic field due to a current I in the wall of a cylindrical tube, assuming the wall to have negligible thickness. Give the field both inside and outside the tube.

3.

Show that the magnetic field intensity in a coaxial cable is given by:



(a) $H = \dfrac{I}{2\pi r} \qquad a \le r \le b$

(b) $H = 0 \qquad r > c$

4.

A long straight conductor carrying a current of 10 A lies along the axis of a thin hollow cylinder of sheet steel. The cylinder has a radius of 20 mm and the steel has a relative permeability of 2000.

Determine the direction and magnitude of the magnetic field intensity, and the flux density, in the steel.

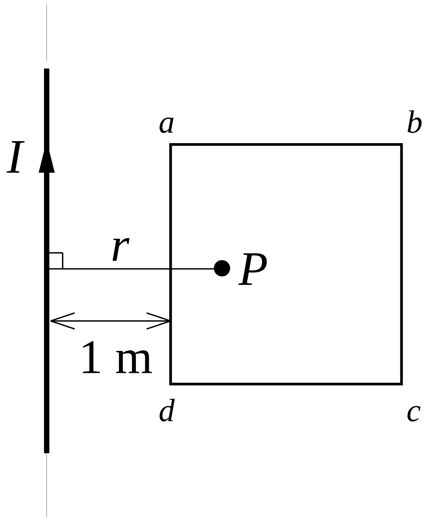
5.

A toroid of square cross-section is manufactured from a sheet steel with a relative permeability of 2000. The inner and outer radii of the toroid are 80 mm and 120 mm respectively. A coil of 500 turns is wound uniformly around the ring and carries a current of 1 mA.

Determine the total magnetic flux in the toroid.

6.

A conducting loop *abcd* lies in a plane containing a long straight current-carrying conductor, as shown below:



The current in the conductor is 10 A. The loop *abcd* is a plane square with sides of length 2 m. The whole system is in air.

(a) Determine the magnetic flux density at a perpendicular distance *r* from the conductor (point *P*), and hence

(b) Calculate the total magnetic flux streaming through the area bounded by the loop.