14 Inductance

Contents

Introduction

The inductor is a circuit element whose voltage-current relationship involves the rate of change of current. Physically, an inductor may be constructed by winding a length of wire into a coil. Energy is stored in the *magnetic field* that exists around the wire in the coil, and the geometry of a coil increases this field compared to a straight wire. In addition, the centre of the coil may be made of a ferromagnetic material (such as iron) which will dramatically increase the inductance (compared to air).

Faraday's and Lenz's Laws will enable us to appreciate the relationships between changing magnetic flux linkage and induced electromotive force.

The inductor is capable of storing and delivering finite amounts of energy, but it cannot deliver non-zero average power over an infinite time interval. It is therefore a *passive* circuit element, like the resistor.

The inductor is a *linear* circuit element. Therefore all the circuit methods previously studied, such as nodal analysis, superposition, Thévenin’s theorem, etc., can be applied to circuits containing inductors.

14.1 The Inductor

An inductor is a two-terminal device, usually constructed as a coil of wire, that is designed to store energy in a magnetic field. The coil effectively increases the magnetic field by the number of "turns" of wire, and as we shall see, also increases the number of "circuits" which "link" the magnetic field.

An ideal inductor produces a magnetic field which is wholly confined within it. The closest approximation to an ideal inductor that we can physically produce is a toroid, which has an almost uniform magnetic field confined within it:

A toroidal inductor

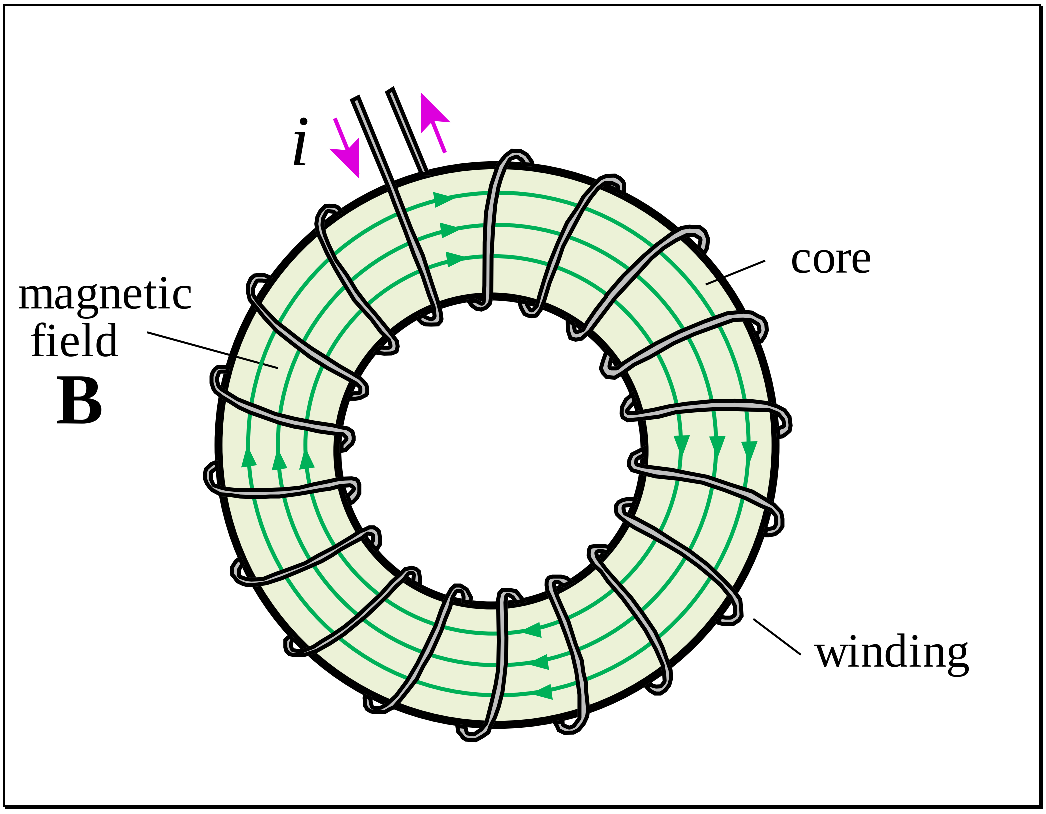


Figure 14.1

14.4

Magnetic flux defined

To explain magnetism, 19th century scientists invoked an analogy with fluids and postulated the existence of a magnetic fluid, known as magnetic "flux", Φ, which streamed throughout space and manifested itself as magnetism. Magnetic flux always streams out of north poles and into south poles, and forms a closed loop. We still use this concept of flux today, as the theory has been spectacularly successful.

If a tube of magnetic flux (which is a closed loop) streams through a closed loop of wire (a circuit), then it is said to "link" with the circuit. The amount of "flux linkage", λ, is given by:

Magnetic flux linkage defined

$$\lambda = N\Phi \tag{14.1}$$

where N is the number of loops of wire, or turns, in the circuit, and Φ is the average amount of magnetic flux streaming through each loop.

Magnetic flux linkage shown graphically

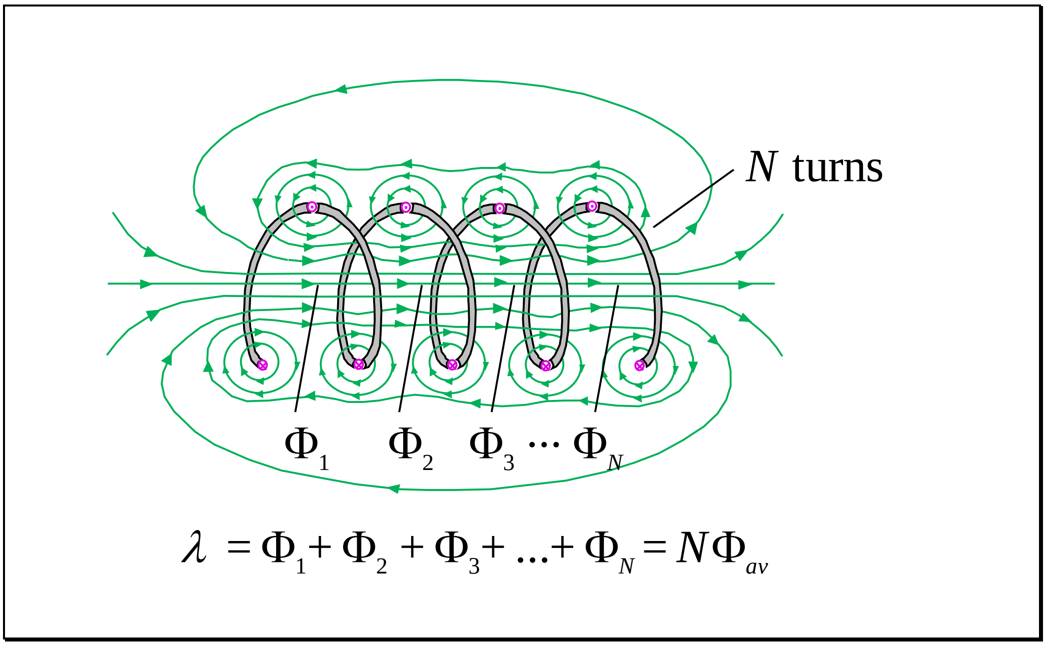


Figure 14.2

An ideal inductor is a structure where the flux linkage (with itself) is directly proportional to the current through it. We define a constant, called the self *inductance*, *L*, of the structure by the linear relationship:

The definition of inductance

$$\lambda = Li \tag{14.2}$$

The unit of inductance is the *henry*, with symbol H.

The ideal inductance relationship is a *straight line through the origin*:

The inductor is a linear circuit element

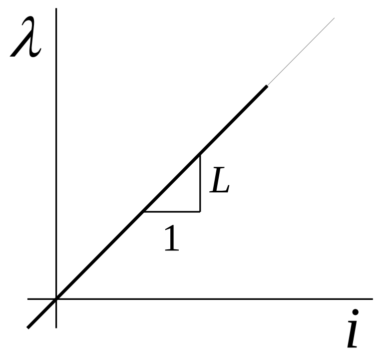


Figure 14.3

Even though inductance is defined as $L = \lambda / i$, it should be noted that L is a purely geometric property, and depends only on the conductor arrangements and the materials used in the construction.

For example, it can be shown for a closely wound toroid that the inductance is approximately:

The inductance of a toroidal inductor

$$L \approx N^2 \frac{\mu A}{l} \tag{14.3}$$

where A is the cross-sectional area of the toroid material, and l is the mean path length around the toroid. The *permeability*, μ, is a constant of the material used in making the toroid. The permeability is usually expressed in terms of *relative permeability*, μ_r:

Relative permeability defined

$$\mu = \mu_r \mu_0 \tag{14.4}$$

where $\mu_0 = 400\pi \text{ nHm}^{-1}$ is the permeability of free space (and, for all practical purposes, air).

14.1.1 Inductor *v-i* Relationships

We now seek a *v-i* relationship for the inductor. In 1840, the great British experimentalist Michael Faraday[1] discovered that a changing magnetic field could induce a voltage in a neighbouring circuit, or indeed the circuit that was producing the magnetic field. Faraday's Law states that the induced voltage is equal to the rate-of-change of magnetic flux linkage:

Faraday's Law

$$v = -\frac{d\lambda}{dt} \tag{14.5}$$

The minus sign comes from the fact that the polarity of the induced voltage is such as to oppose the change in flux. For an inductor, we can figure out that the polarity of the induced voltage must be positive at the terminal where the current enters the inductor. If we know this, then we can mark the polarity on a circuit diagram and deal with the magnitude of the induced voltage by dropping the minus sign (the determination of the voltage polarity is called Lenz's Law).

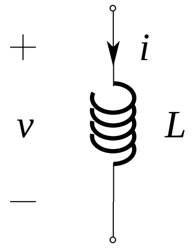
If we allow the polarity to be set by Lenz's Law, and substitute $\lambda = Li$ into the previous equation, then we get:

The inductor's *v-i* branch derivative relationship

$$v = L\frac{di}{dt} \tag{14.6}$$

[1] The American inventor Joseph Henry discovered this phenomenon independently, but Faraday was the first to publish.

The circuit symbol for the inductor is based on the construction of the physical device, and is shown below together with the passive sign convention for the voltage and current:



The circuit symbol for the inductor

Figure 14.4

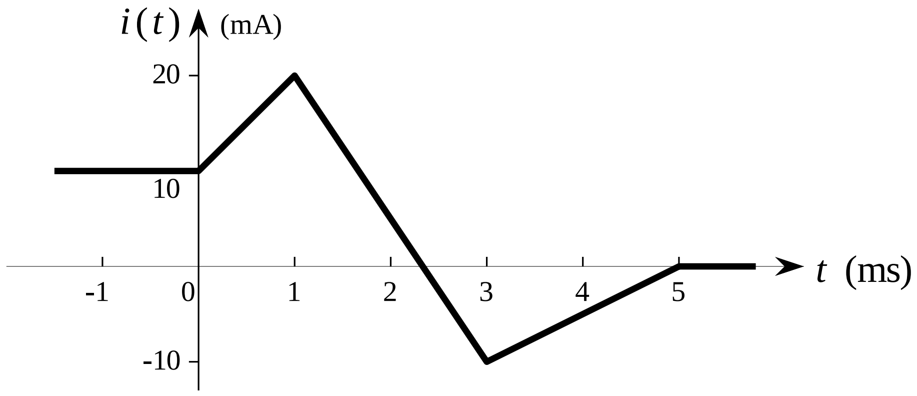
The inductor current may be expressed in terms of the voltage by integrating Eq. (14.6):

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v\,d\tau + i(t_0) \tag{14.7}$$

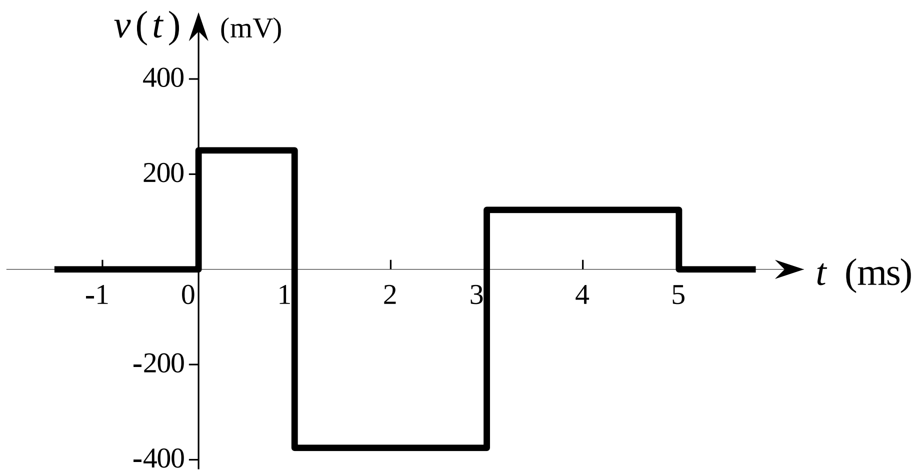
The inductor's v-i branch integral relationship

EXAMPLE 14.1 Inductor Current and Voltage Waveforms

The variation of current through a 25 mH inductor as a function of time is shown below:

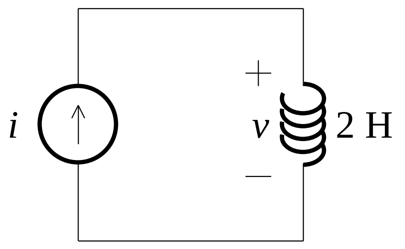


Since the current is constant for $t < 0$, the voltage is zero in this interval. The current then begins to increase at the linear rate $di/dt = 10\ \text{As}^{-1}$, and thus a constant voltage of $v = L\,di/dt = 250\,\text{mV}$ is produced. During the following 2 millisecond interval, the current decreases at the linear rate $di/dt = -15\ \text{As}^{-1}$, and so the voltage is $v = L\,di/dt = -375\,\text{mV}$. The final increase of the current causes a positive 125 mV and no response thereafter. The voltage waveform is sketched below:

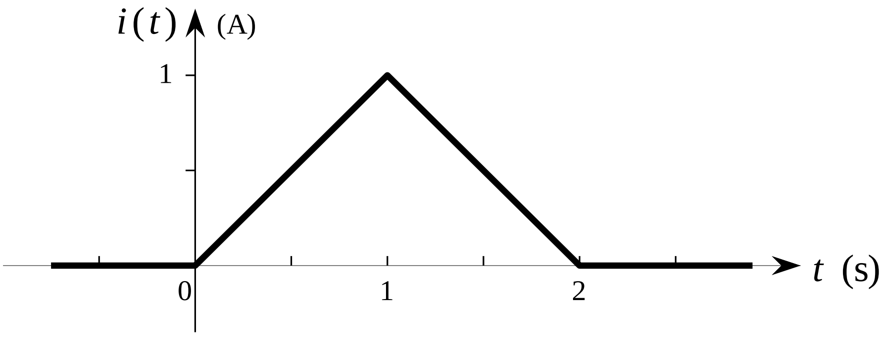


EXAMPLE 14.2 Inductor Current, Voltage and Power Waveforms

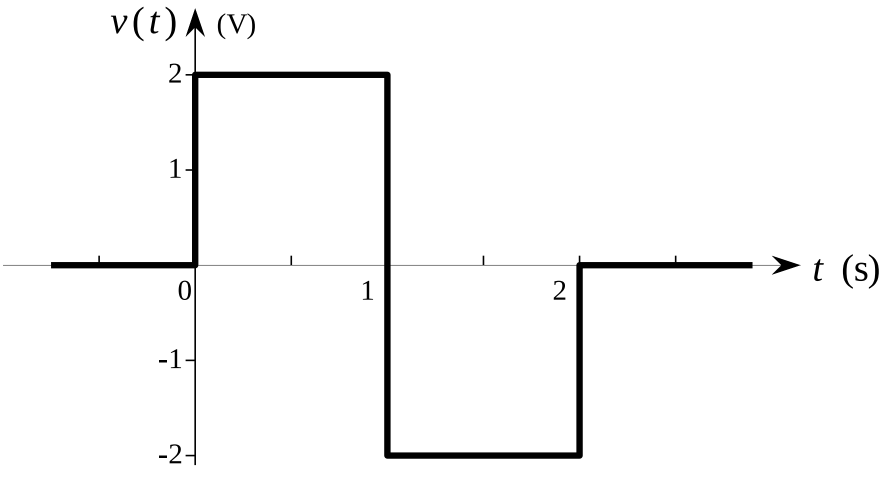
Consider the simple circuit below:



Let the current $i(t)$ which is produced by the source be described by the function of time shown below:

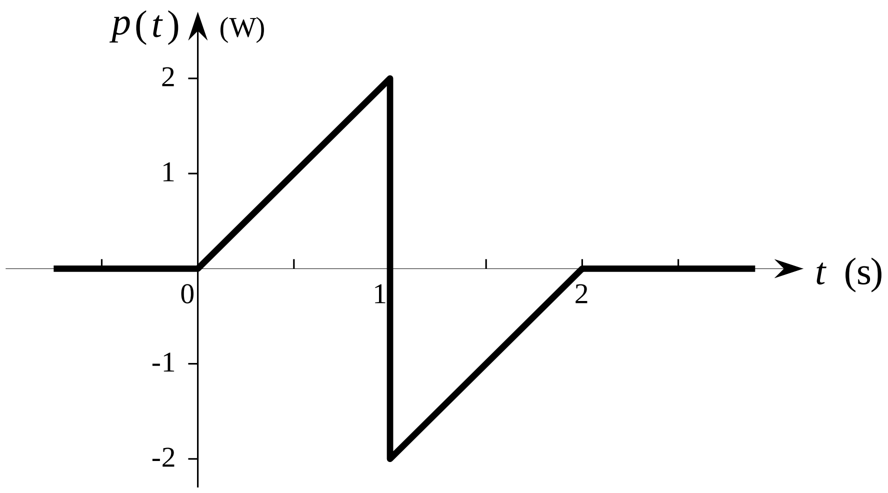


Since $v = L\,di/dt = 2\,di/dt$ then the voltage across the inductor is as shown below:



14.10

We know that a resistor always absorbs power and the energy absorbed is dissipated as heat – but how about an inductor? For the inductor in this example, the instantaneous power $p(t) = v(t)i(t)$ absorbed by the inductor is given by the graph below:



We see that the power absorbed by the inductor is zero for $-\infty < t \le 0$ and $2 \le t \le \infty$. For $0 < t < 1$, since $p(t)$ is a positive quantity, the inductor is absorbing power (which is produced by the source). However, for $1 < t < 2$, since $p(t)$ is a negative quantity, the inductor is actually supplying power (to the source).

To get the energy absorbed by the inductor, we simply integrate the power absorbed over time. For this example, the energy absorbed increases from 0 to $\frac{1}{2}(1)(2) = 1\text{ J}$ as time goes from $t = 0$ to $t = 1\text{ s}$. However, from $t = 1$ to $t = 2\text{ s}$, the inductor supplies energy such that at time $t = 2\text{ s}$ and thereafter, the net energy absorbed by the inductor is zero. Since all of the energy absorbed by the inductor is not dissipated but is eventually returned, we say that the inductor *stores* energy. The energy is stored in the magnetic field that surrounds the inductor.

14.1.2 Energy Stored in an Inductor

The power delivered to an inductor is:

$$p = vi = L\frac{di}{dt}i \qquad (14.8)$$

and the energy stored in its magnetic field is therefore:

$$\begin{aligned} w_L(t) &= \int_{t_0}^{t} p\,dt + w_L(t_0) \\ &= L\int_{t_0}^{t} i\frac{di}{dt}dt + w_L(t_0) \\ &= L\int_{i(t_0)}^{i(t)} i\,di + w_L(t_0) \\ &= \frac{1}{2}L\left[i^2(t) - i^2(t_0)\right] + w_L(t_0) \end{aligned} \qquad (14.9)$$

If the inductor current is zero at t_0, then the magnetic field, and hence the stored inductor energy, $w_L(t_0)$, is also zero at that instant. We then have:

$$w_L(t) = \frac{1}{2}Li^2(t) \qquad (14.10)$$

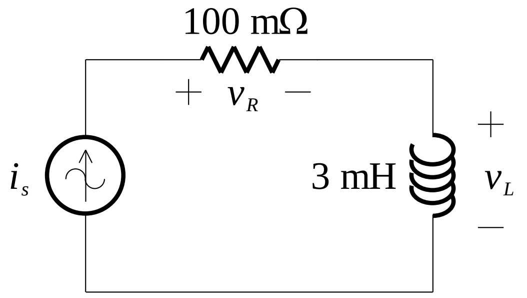
The stored energy in an inductor

We can see that the energy stored in an inductor depends only on the inductance and the current. Therefore, a finite amount of energy can be stored in an inductor even if the voltage across the inductor is zero.

Whenever the current is not zero, and regardless of its direction, energy is stored in the inductor. It follows, therefore, that power must be delivered to the inductor for a part of the time and recovered from the inductor later.

EXAMPLE 14.3 Energy Stored in an Inductor

A sinusoidal current source is connected in series with a $100\ \text{m}\Omega$ resistor and a 3 mH inductor, as shown below:



The $100\ \text{m}\Omega$ resistor in the circuit represents the resistance of the wire which must be associated with the physical coil. Let $i_s = 12\sin(100\pi t)$ A .

The voltage across the resistor is:

$$v_R = Ri = 1.2\sin(100\pi t) \quad \text{V}$$

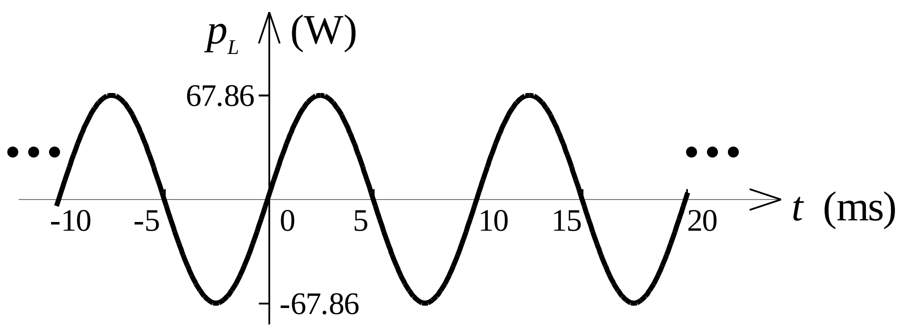
and the voltage across the inductor is:

$$v_L = L\frac{di}{dt} = 3\times10^{-3}\frac{d}{dt}[12\sin(100\pi t)] = 11.31\cos(100\pi t) \quad \text{V}$$

The power delivered to the inductor is:

$$p_L = v_L i = 11.31\cos(100\pi t)\times 12\sin(100\pi t) = 67.86\sin(200\pi t) \quad \text{W}$$

A graph of the power delivered to the inductor versus time is shown below:

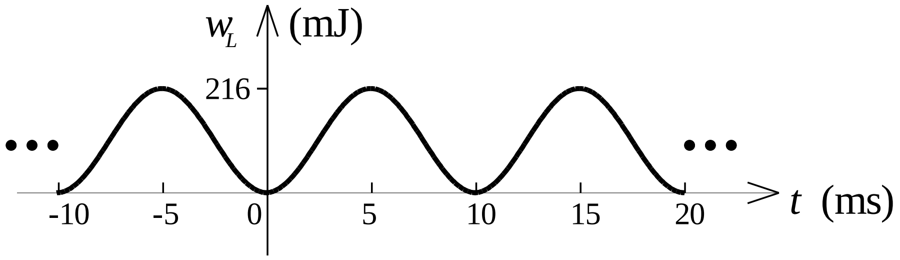


The *instantaneous* power delivered to the inductor is sinusoidal, and varies at *twice* the frequency of the current source. Negative power *delivered to* the inductor means that power is *sourced from* the inductor. Note that the *average* power dissipated in the inductor is zero.

The energy stored in the inductor is:

$$w_L = \frac{1}{2} L i^2 = \frac{1}{2} \times 3 \times 10^{-3} [12 \sin(100\pi t)]^2 = 216 \sin^2(100\pi t) \quad \text{mJ}$$

A graph of the stored energy versus time is shown below:



The energy stored is sinusoidal and varies at twice the frequency of the current source, but it also has a finite *average* component of 108 mJ.

We see that the energy increases from zero at $t = 0$ to a maximum of 216 mJ at $t = 5\ \text{ms}$ and then decreases to zero in another 5 ms. During this 10 ms interval, the energy dissipated in the resistor as heat is:

$$w_R = \int_0^{0.01} p_R dt = \int_0^{0.01} 14.4 \sin^2(100\pi t) dt = \int_0^{0.01} 7.2[1 - \cos(200\pi t)] dt = 72 \quad \text{mJ}$$

Thus, an energy equal to 33.33% of the maximum stored energy is lost as heat in the process of storing and removing the energy in the physical inductor.

14.1.3 Summary of Important Inductor Characteristics

Several important characteristics of the ideal inductor are summarised below.

The inductor is a short-circuit to DC

1. If a constant current is passed through an inductor, then $di/dt = 0$ and subsequently no voltage appears across it. An inductor is thus **a short-circuit to DC**.

An inductor current cannot change instantaneously

2. An inductor **current cannot change instantaneously**, for this implies $di/dt = \infty$, and the inductor would require infinite voltage. Thus, inductor current is smooth and continuous. This fact will be used frequently when undertaking *transient analysis* of circuits.

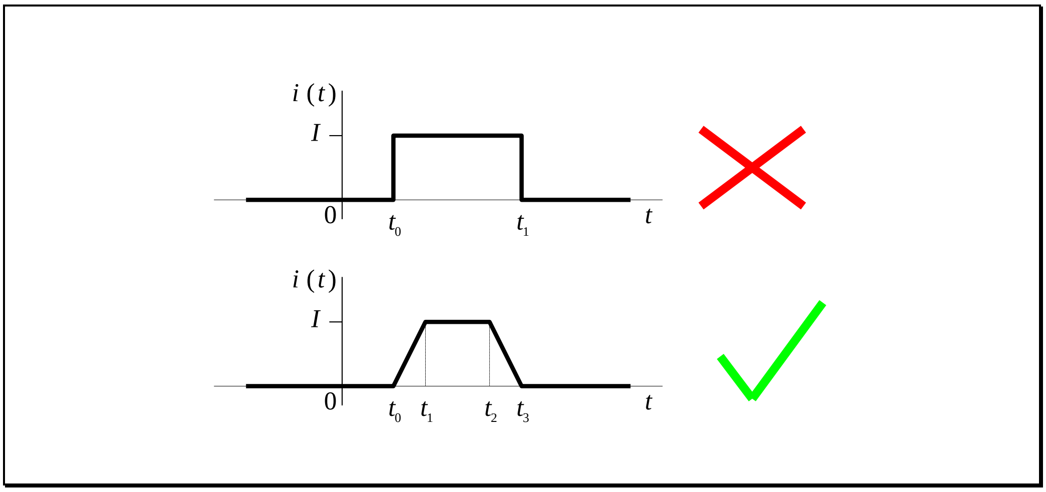


Figure 14.5

3. For an ideal inductor, **voltage can change instantaneously**.

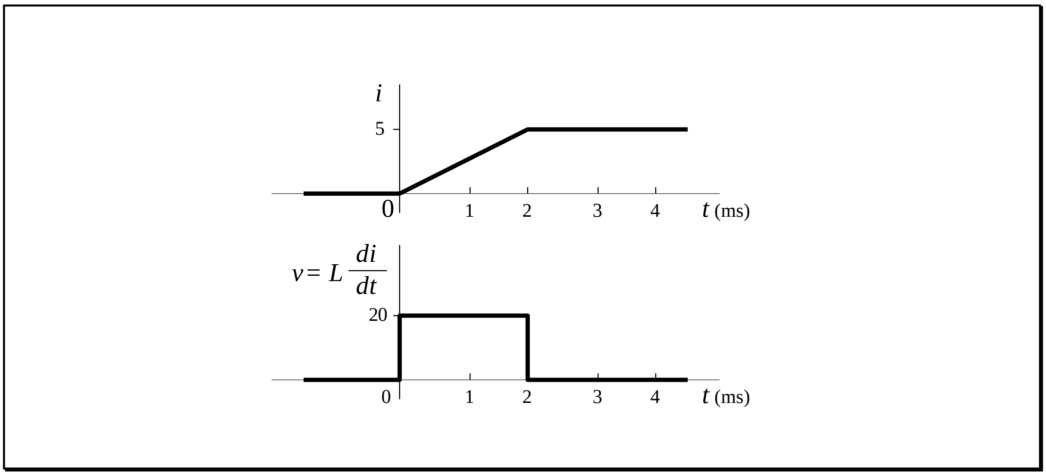


Figure 14.6

4. An ideal inductor never dissipates energy, but stores it and releases it using its magnetic field.

14.2 Practical Inductors

Practical inductors are manufactured for different values, voltage and current ratings, accuracy, volumetric efficiency, temperature stability, etc. As such, their construction will play a role in their electrical behaviour. To represent these physical components in a circuit we need to model their non-ideal and parasitic effects.

Practical components have non-ideal and parasitic effects

There are many different types of inductor construction. Some are shown below, labelled by the type of core:

ferrite core

laminated steel core

air core

Some types of inductors

Figure 14.7 – Some types of inductors

The construction of the *core* of the inductor is especially important in determining its properties. Laminated steel cores are common for low frequencies, and ferrites (non-conducting oxides of iron) are used for high frequencies. Since the core is subjected to a changing magnetic field, the induced voltages in the core create what are known as *eddy currents*. Steel laminations and ferrites reduce the energy losses caused by these eddy currents. There are also energy loss mechanisms associated with reversing the “magnetic domains” in a core, and so the losses due to the magnetic characteristic and the induced eddy currents are combined into a term called the *core loss*.

The core of an inductor is generally a magnetic material which exhibits energy losses

Air-cored inductors are linear and do not exhibit core losses (since there is no conductive core). They can be made by winding a coil on a non-magnetic former, such as plastic, or may be self-supporting if made large enough. Air-cored inductors have lower inductance than ferromagnetic-core inductors, but are often used at high frequencies because they are free of core losses.

Air-cored inductors do not exhibit core losses

A reasonably accurate circuit model of a real inductor is shown below:

Model of a real inductor

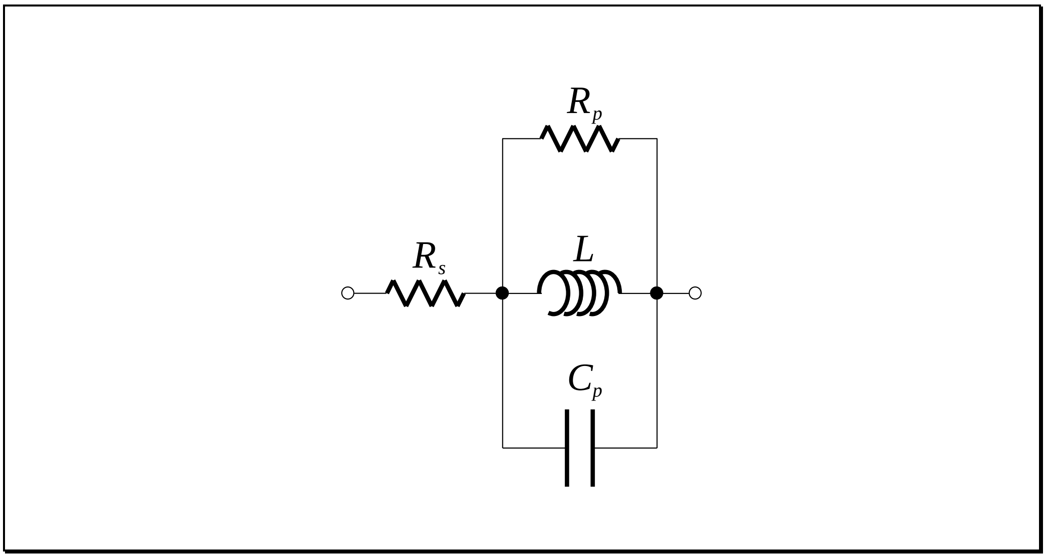


Figure 14.8

The series resistance R_s takes into account the finite resistance of the wire used to create the coil. The parallel capacitance C_p is associated with the electric field in the insulation surrounding the wire, and is called *interwinding capacitance*. The parallel resistance R_p represents the core losses.

The following inductor model, showing just the predominant non-ideal effect of finite winding resistance, is often used at low frequencies:

Low frequency model of a real inductor showing the winding resistance

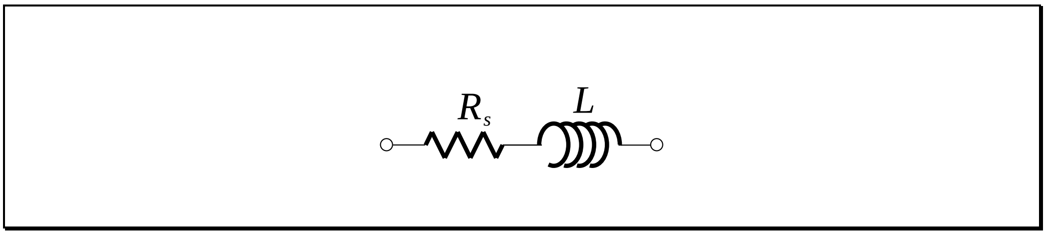


Figure 14.9

14.3 Series and Parallel Connections of Inductors

Inductors connected in series and parallel can be combined into an equivalent inductor. A series connection of inductors is shown below:

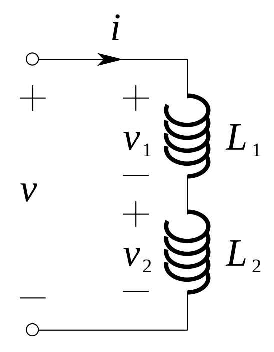


Figure 14.10

We find from KVL that:

$$\begin{aligned} v &= v_1 + v_2 \\ &= L_1 \frac{di}{dt} + L_2 \frac{di}{dt} \\ &= (L_1 + L_2) \frac{di}{dt} \\ &= L \frac{di}{dt} \end{aligned} \qquad (14.11)$$

where:

$$L = L_1 + L_2 \qquad \text{(series)} \qquad (14.12)$$

Combining inductors in series

We depict this below:

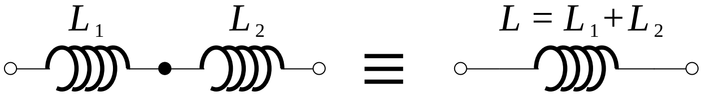


Figure 14.11

14.18

For the case of two inductors in parallel, as shown below:

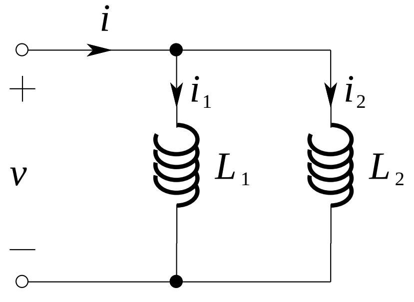


Figure 14.12

we have, by KCL:

$$
\begin{aligned}
i &= i_1 + i_2 \\
&= \frac{1}{L_1}\int_{t_0}^{t} v\,dt + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v\,dt + i_2(t_0) \\
&= \left(\frac{1}{L_1} + \frac{1}{L_2}\right)\int_{t_0}^{t} v\,dt + i_1(t_0) + i_2(t_0) \\
&= \frac{1}{L}\int_{t_0}^{t} v\,dt + i(t_0)
\end{aligned}
\tag{14.13}
$$

where:

Combining inductors in parallel

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2} \qquad \text{(parallel)} \tag{14.14}$$

and:

$$i(t_0) = i_1(t_0) + i_2(t_0) \tag{14.15}$$

We depict this below:

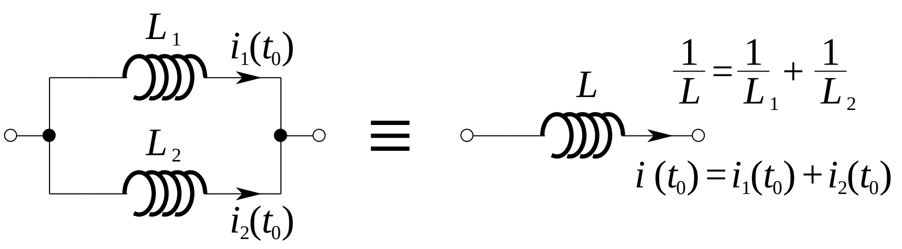


Figure 14.13

14.4 Circuit Analysis with Inductors

For the ideal inductor, the relationships between voltage and current is a linear relationship (apart from the integral relationship that have an initial condition term). Consequently, circuit analysis techniques which rely on the linearity property (such as nodal analysis, mesh analysis, superposition, Thévenin's theorem and Norton's theorem) can be applied to circuits containing inductors.

Linear circuit analysis techniques can be applied to circuits with inductors

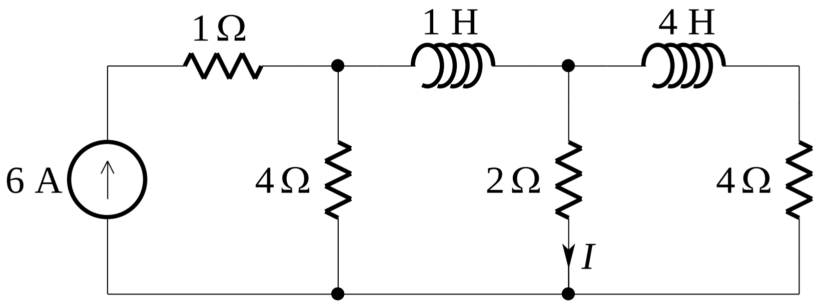
14.4.1 DC Circuits

An ideal inductor behaves as a short-circuit to DC. We can use this fact to determine *steady-state* voltages and currents in DC circuits that contain inductors. Other sources, such as sinusoidal sources, will be treated later.

A DC circuit is treated as a purely resistive circuit in the "steady-state"

EXAMPLE 14.4 DC Analysis of a Circuit with Inductors

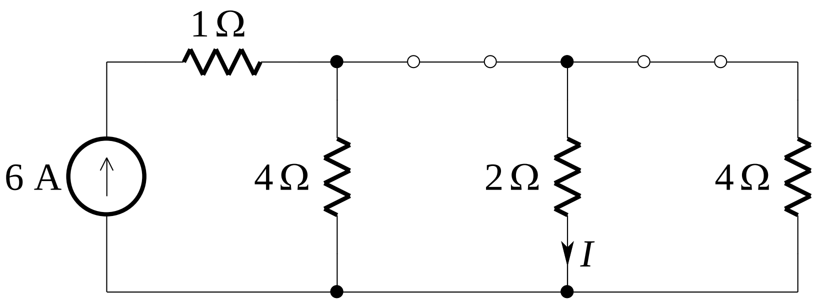
Determine the steady-state direct current I in the circuit below:



The circuit has one independent current source whose value is constant. For a resistive circuit we would naturally anticipate that all voltages and currents are constant. However, this is not a resistive circuit. Yet, our intuition suggests that the constant-valued current source produces constant-valued responses. This fact will be confirmed more rigorously later. In the meantime, we shall use the result that a circuit containing only constant-valued sources is a DC circuit.

14.20

Since for DC all inductors behave like short-circuits, we can replace the original circuit with an equivalent resistive circuit:



By current division we find the steady-state direct current as:

$$i = \frac{2}{2+2}(6) = 3\text{ A}$$

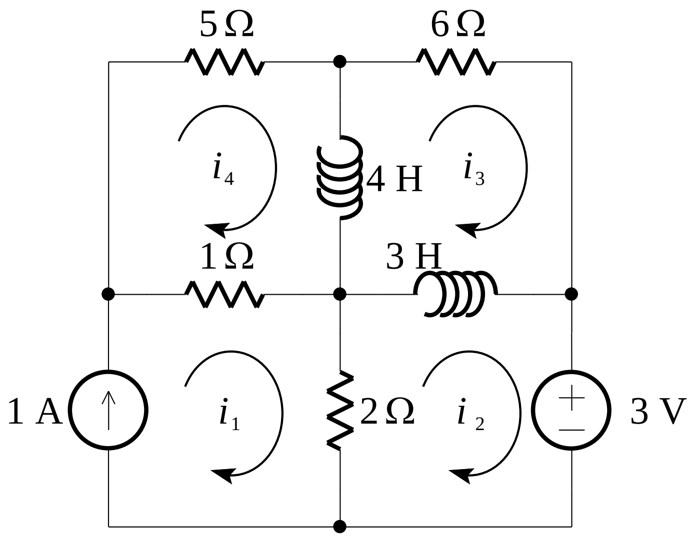
14.4.2 Nodal and Mesh Analysis

Just as we analysed resistive circuits with the use of node and mesh equations, we can write a set of equations for circuits that contain inductors in addition to resistors and sources. The procedure is similar to that described for the resistive case – the difference being that for inductors the appropriate relationship between voltage and current is used in place of Ohm's Law.

Nodal and mesh analysis can be applied to circuits with inductors

EXAMPLE 14.5 Mesh Analysis of a Circuit with Inductors

Consider the circuit shown below:



By mesh analysis, for mesh i_1:

$$i_1 = 1$$

For mesh i_2:

$$2(i_2 - i_1) + 3\frac{d}{dt}(i_2 - i_3) = -3$$

For mesh i_3:

$$3\frac{d}{dt}(i_3 - i_2) + 4\frac{d}{dt}(i_3 - i_4) + 6i_3 = 0$$

For mesh i_4:

$$1(i_4 - i_1) + 5i_4 + 4\frac{d}{dt}(i_4 - i_3) = 0$$

The equations obtained for meshes i_2, i_3 and i_4 are called *differential equations* since they contain variables and their derivatives.

Writing the equations for a circuit, as in the preceding example, is not difficult. Finding the solution of equations like these, however, is another matter – it is no simple task. Thus, with the exception of some very simple circuits, we shall have to resort to additional concepts and techniques to be introduced later.

14.5 Summary

- Inductance is defined as the ratio of flux linkage to current of a structure:

$$\lambda = Li$$

- The inductance of a toroid is:

$$L = N^2 \frac{\mu A}{l} = N^2 \frac{\mu_r \mu_0 A}{l}$$

where A is the cross-sectional area of the toroid, l is the mean path length of the toroid, and the relative permeability μ_r is a property of the toroidal material (core).

- The fundamental relationship between magnetic flux density and magnetic field intensity is:

$$\mathbf{B} = \mu \mathbf{H}$$

- The *v-i* relationship for an inductor is:

$$v = L \frac{di}{dt}$$

- An inductor behaves like a **short-circuit** to **direct current** in the steady-state.

- The current through an inductor cannot change instantaneously.

- The energy stored in an inductor is:

$$w_L = \frac{1}{2} Li^2$$

- Inductors in series and parallel are combined in the same way as are resistances.

- Writing node and mesh equations for circuits containing inductors is done in the same manner as for resistive circuits. Obtaining solutions of equations in this form will be avoided, except for simple circuits.

14.6 References

Bobrow, L.: *Elementary Linear Circuit Analysis*, Holt-Saunders, 1981.

Hayt, W. & Kemmerly, J.: *Engineering Circuit Analysis*, 3rd *Ed.*, McGraw-Hill, 1984.

Exercises

1.

There is a current $i = 5\sin 10t\text{ A}$ through an inductance $L = 2\text{ H}$. What is the first instant of time after $t = 0$ when the power entering the inductor is exactly:

(a) 100 W

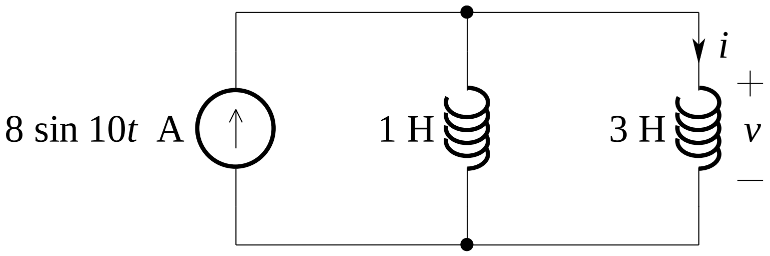
(b) -100 W

2.

The energy stored in a certain 10 mH inductor is zero at $t = 1\text{ ms}$ and increases linearly by 20 mJ each second thereafter. Find the inductor current and voltage for $t > 1\text{ ms}$ if neither is ever negative.

3.

At $t = 0$, $i = 5\text{ A}$ in the circuit shown below:



(a) Find $v(t)$ for all t.

(b) Find $i(t)$ for $t \geq 0$.