20 The Phasor Concept

Contents

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Introduction

A sinusoidal voltage or current *at a given frequency* is characterized by only two parameters, an amplitude and a phase.

A complex number "carries" two pieces of information – a magnitude and an angle.

If we make the claim that there is a *correspondence* between sinusoids and complex numbers such that:

$$\text{amplitude} \leftrightarrow \text{magnitude}$$

$$\text{phase} \leftrightarrow \text{angle}$$

then we can formalise AC circuit analysis by operating with complex numbers instead of sinusoids.

Thus, we will need to represent voltages and currents in an AC circuit with a complex representation characterized by a magnitude and an angle. We will also need to represent resistors, capacitors and inductors as complex numbers in order for the usual voltage-current relationships to hold.

20.1 The Sinusoidal Steady-State Response

Let's firstly consider what seems to be a simple problem, but one which turns out to be cumbersome to solve. If we apply a sinusoidal voltage to a circuit, we obtain (after a time[1]) a "steady-state" condition in which all voltages and currents are sinusoids, equal in frequency to the source, but differing in both amplitude and phase.

The sinusoidal steady-state response defined

In an AC circuit we therefore expect that all voltages and currents will be sinusoidal eventually. The term *steady-state* is used synonymously with the "response" of a circuit after a *transient* period. The circuits we are about to analyse are commonly said to be in the "sinusoidal steady-state". Unfortunately, steady-state implies "not changing with time", but this is not correct – the sinusoidal forced response definitely changes with time. The steady-state simply refers to the condition which is reached after the transient response has died out.

Consider the series *RL* circuit below:

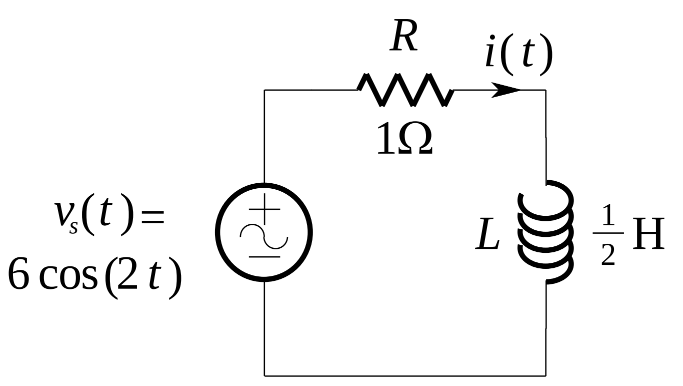


Figure 20.1 – A series *RL* circuit with a sinusoidal source

[1] "after a time" means the circuit will need to change from its current state (perhaps all zero voltages and currents) to its final state – when it does this, all voltages and currents will experience what is termed a *transient*. However, the circuit will eventually "settle down" such that all voltages and currents are sinusoids. Transients will be explored in future study.

20.4

The sinusoidal source voltage $v_s = 6\cos 2t$ has been switched into the circuit at some remote time in the past, and any transient response has died out completely.

We seek the sinusoidal steady-state response. Applying KVL around the loop, and using the fundamental branch relationships for the elements, we get:

$$L\frac{di}{dt} + Ri = v_s \tag{20.1}$$

Making the equation "monic" (coefficient of the derivative is one), we get:

$$\frac{di}{dt} + \frac{R}{L}i = \frac{v_s}{L} \tag{20.2}$$

Substituting values:

$$\frac{di}{dt} + 2i = 12\cos 2t \tag{20.3}$$

This equation is quite difficult to solve with our current mathematical tools, so we will take a more "intuitive" approach to the problem. We assumed that all voltages and currents (in the steady-state) are sinusoids, so we should be able to express the current as a single sinusoid with an unknown amplitude and phase angle. We will therefore let:

$$i(t) = A\cos(2t + \theta) \tag{20.4}$$

In order to avoid using the trigonometric identity for the cosine of the sum of two angles, we will use the alternative form:

$$i(t) = A_1\cos 2t + A_2\sin 2t \tag{20.5}$$

Substituting this into Eq. (20.3) results in:

$$\left(-2A_1 \sin 2t + 2A_2 \cos 2t\right) + 2\left(A_1 \cos 2t + A_2 \sin 2t\right) = 12\cos(2t) \tag{20.6}$$

or, upon grouping cos and sin terms:

$$\left(-2A_1 + 2A_2\right)\sin 2t + \left(2A_1 + 2A_2\right)\cos 2t = 12\cos(2t) \tag{20.7}$$

Equating like coefficients of $\cos 2t$ and $\sin 2t$, we get the simultaneous equations:

$$\begin{aligned} -2A_1 + 2A_2 &= 0 \\ 2A_1 + 2A_2 &= 12 \end{aligned} \tag{20.8}$$

The solution to these equations is $A_1 = 3$ and $A_2 = 3$. Hence, the steady-state current is:

$$i(t) = 3\cos 2t + 3\sin 2t \tag{20.9}$$

or:

$$i(t) = 3\sqrt{2}\cos\left(2t - \frac{\pi}{4}\right) \tag{20.10}$$

The steady-state current for the series *RL* circuit to a sinusoidal voltage source

20.6

The steady-state response graphed, showing only an amplitude and phase change

A plot of the functions $v_s(t)$ and $i(t)$ versus t is shown below:

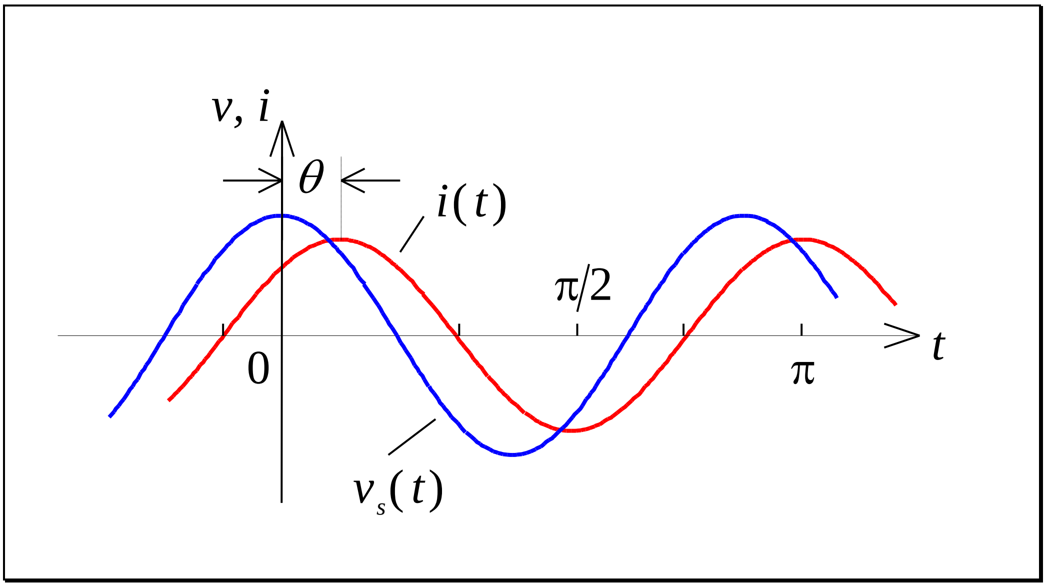


Figure 20.2 – Voltage and current for a series *RL* circuit

Since $v_s(t)$ reaches its peak before $i(t)$, we can say that $v_s(t)$ *leads* $i(t)$ by $\pi/4$ radians (or $45°$) or that $i(t)$ *lags* $v_s(t)$ by $\pi/4$ radians (or $45°$).

The fact that current lags the voltage in this simple *RL* circuit is now visually apparent. Note also that the frequency of the response is the same as the source.

The voltage across the inductor is:

$$
\begin{aligned}
v_L &= L\frac{di}{dt} = \frac{1}{2}\frac{di}{dt} \\
&= \frac{1}{2}\left(-6\sin 2t + 6\cos 2t\right) \\
&= -3\sin 2t + 3\cos 2t \\
&= 3\sqrt{2}\cos\left(2t + \frac{\pi}{4}\right)
\end{aligned}
\tag{20.11}
$$

Thus, the voltage $v_L(t)$ leads the voltage $v_s(t)$ by $45°$ and $i(t)$ lags $v_L(t)$ by $90°$.

The method by which we found the sinusoidal steady-state response for the simple *RL* circuit is quite intricate. It would be impractical to analyse every circuit by this method. We shall see in the next section that there is a way to simplify the analysis. It involves the formulation of complex algebraic equations instead of differential equations, but the advantage is that we can produce a set of complex algebraic equations for a circuit of any complexity. Sinusoidal steady-state analysis becomes almost as easy as the analysis of resistive circuits.

20.2 The Complex Forcing Function

It seems strange at first, but the use of complex quantities in sinusoidal steady-state analysis leads to methods which are simpler than those involving only real quantities.

Consider a sinusoidal source:

$$V_m \cos(\omega t + \theta) \tag{20.12}$$

which is connected to a general, passive, linear, time-invariant (LTI) circuit as shown below:

Excitation of a passive LTI circuit by a real sinusoid produces a real sinusoidal response

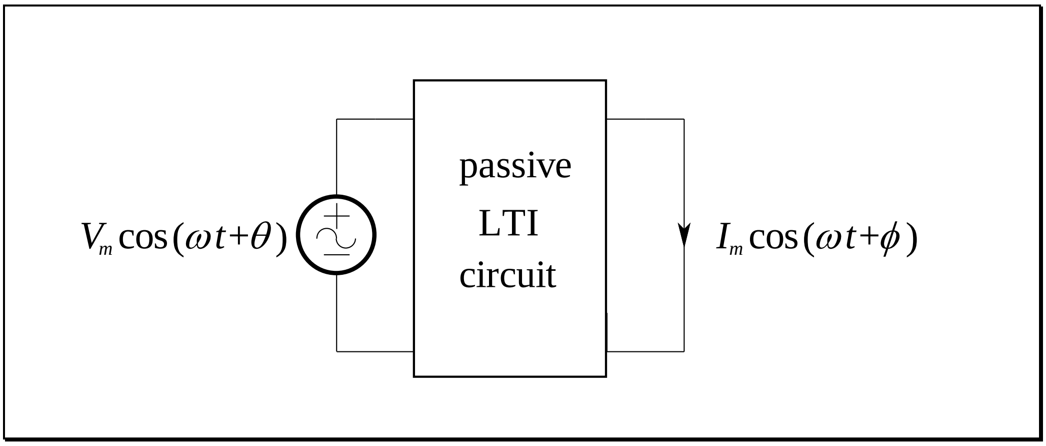


Figure 20.3 – A real source produces a real response

A current response in some other branch of the circuit is to be determined, and we know for a sinusoidal voltage source that the current is sinusoidal. Let the sinusoidal current be represented by:

$$I_m \cos(\omega t + \phi) \tag{20.13}$$

Note that the frequency stays the same – only the amplitude and phase are unknown.

If we delay the forcing function by $90°$, then since the system is time-invariant, the corresponding forced response must be delayed by $90°$ also (because the frequencies are the same). Thus, the voltage source:

$$V_m \cos(\omega t + \theta - 90°) = V_m \sin(\omega t + \theta) \tag{20.14}$$

will produce a current:

$$I_m \cos(\omega t + \phi - 90°) = I_m \sin(\omega t + \phi) \tag{20.15}$$

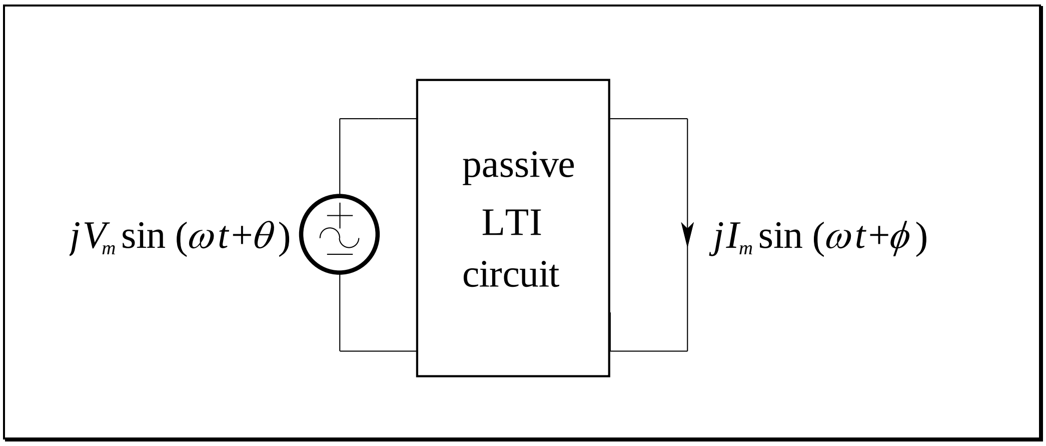
Since the circuit is linear, if we double the source, we double the response. In fact, if we multiply the source by any constant *k*, we achieve a response which is *k* times bigger. We now construct an imaginary source – we multiply the source by $j = \sqrt{-1}$. We thus apply:

$$jV_m \sin(\omega t + \theta) \tag{20.16}$$

and the response is:

$$jI_m \sin(\omega t + \phi) \tag{20.17}$$

The imaginary source and response are shown below:



Excitation of a passive LTI circuit by an imaginary sinusoid produces an imaginary sinusoidal response

Figure 20.4 – An imaginary source produces an imaginary response

20.10

We have applied a real source and obtained a real response, and we have applied an imaginary source and obtained an imaginary response. We can now use the superposition theorem (the circuit is linear) to find the response to a complex *forcing function* which is the sum of the real and imaginary voltage sources. Thus, the sum of the voltage sources of Eqs. (20.12) and (20.16) is:

$$V_m \cos(\omega t + \theta) + jV_m \sin(\omega t + \theta) \tag{20.18}$$

and it produces a response which is the sum of Eqs. (20.13) and (20.17):

$$I_m \cos(\omega t + \phi) + jI_m \sin(\omega t + \phi) \tag{20.19}$$

The complex source and response may be represented more simply by applying Euler's identity:

Euler's identity

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{20.20}$$

Thus, the complex forcing function which is equivalent to Eq. (20.18) is:

$$V_m e^{j(\omega t + \theta)} \tag{20.21}$$

and it produces a response which is equivalent to Eq. (20.19):

$$I_m e^{j(\omega t + \phi)} \tag{20.22}$$

The complex source and response are illustrated below:

Excitation of a passive LTI circuit by a complex source produces a complex response

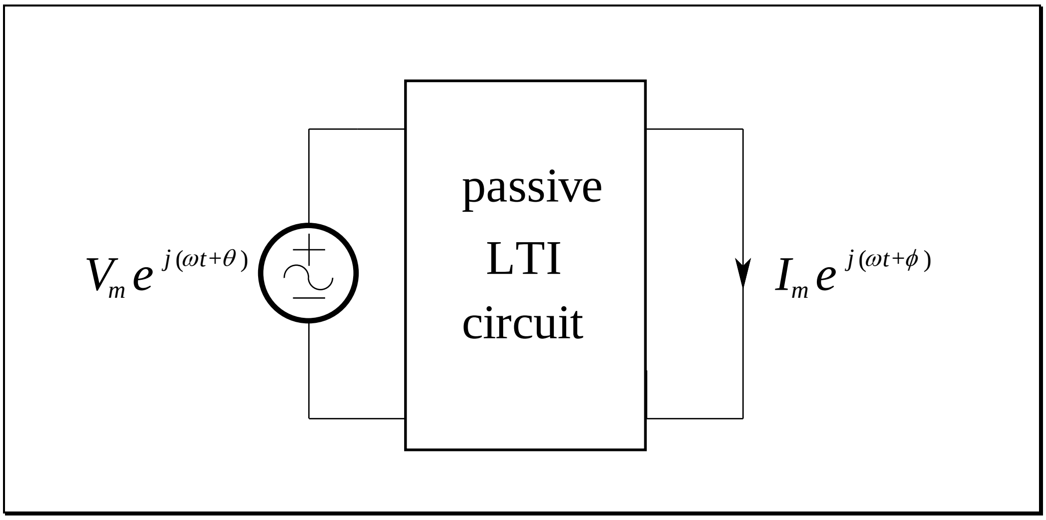


Figure 20.5 – A complex source produces a complex response

We are now ready to see how this helps with sinusoidal analysis. We first note that the real part of the complex response is produced by the real part of the complex forcing function, and the imaginary part of the complex response is produced by the imaginary part of the complex forcing function.

Our strategy for sinusoidal analysis will be to apply a complex forcing function whose real part is the given real forcing function – we should then obtain a complex response whose real part is the desired real response.

We analyse circuits in the sinusoidal steady-state by using a complex forcing function whose real part is the given real forcing function

We will try this strategy on the previous *RL* circuit:

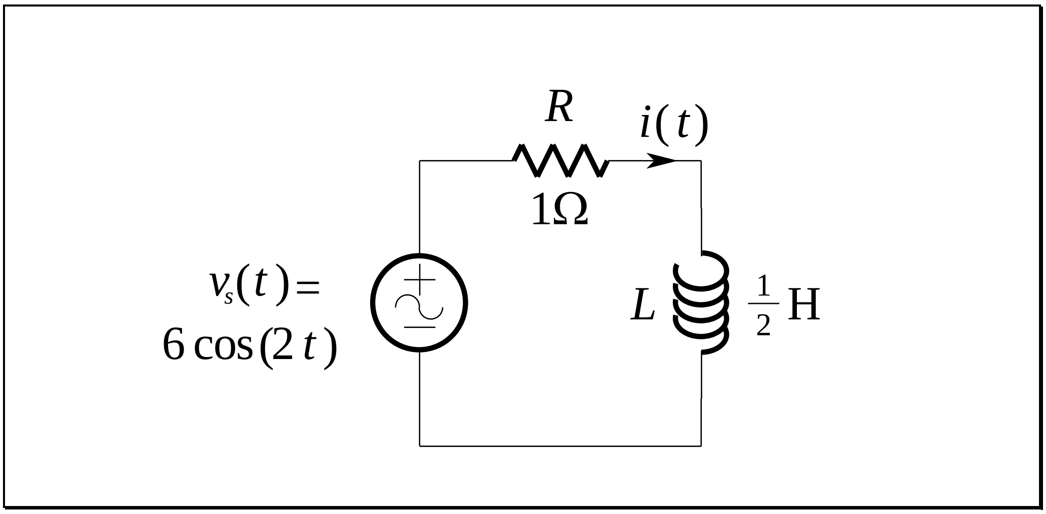


Figure 20.6 – A series *RL* circuit with a sinusoidal source

The real source $v_s = 6\cos 2t$ is applied, and the real response $i(t)$ is desired.

We first construct the complex forcing function by adding an appropriate imaginary component to the given real forcing function. The necessary complex source is:

$$6\cos 2t + j6\sin 2t = 6e^{j2t} \tag{20.23}$$

The complex response which results is expressed in terms of an unknown amplitude I_m and an unknown phase angle ϕ:

$$I_m e^{j(2t+\phi)} \tag{20.24}$$

Writing the differential equation for this circuit:

$$\frac{di}{dt} + 2i = 2v_s \tag{20.25}$$

we insert our complex expressions for v_s and i:

$$\frac{d}{dt}\left(I_m e^{j(2t+\phi)}\right) + 2I_m e^{j(2t+\phi)} = 12e^{j2t} \tag{20.26}$$

Taking the indicated derivative gives:

Using complex sources and responses reduces the original differential equation to a complex algebraic equation

$$j2I_m e^{j(2t+\phi)} + 2I_m e^{j(2t+\phi)} = 12e^{j2t} \tag{20.27}$$

which is a complex *algebraic* equation. This is a considerable advantage – we have turned a *differential* equation into an *algebraic* equation. The only "penalty" is that the algebraic equation uses complex numbers. It will be seen later that this is not a significant disadvantage.

In order to determine the value of I_m and ϕ, we divide through by the common factor e^{j2t}, and we'll place the imaginary part after the real part on the left-hand side so that it looks more like a complex number in rectangular notation:

$$2I_m e^{j\phi} + j2I_m e^{j\phi} = 12 \tag{20.28}$$

Removing the common factor of 2 and factoring the left side gives:

$$(1+j)I_m e^{j\phi} = 6 \tag{20.29}$$

Rearranging, we have:

$$I_m e^{j\phi} = \frac{6}{1+j1} \tag{20.30}$$

The complex response expressed in rectangular form

This is the complex response, and it was obtained in a few easy steps. If we express the response in exponential or polar form, we have:

$$I_m e^{j\phi} = 3\sqrt{2}e^{-j\pi/4} \tag{20.31}$$

The complex response expressed in polar form

Thus, by comparison:

$$I_m = 3\sqrt{2} \tag{20.32}$$

and:

$$\phi = -\pi/4 \tag{20.33}$$

We said that the complex response was:

$$I_m e^{j(2t+\phi)} = I_m e^{j\phi} e^{j2t} \tag{20.34}$$

and that the real response was just the real part of the complex response. Therefore, the real response $i(t)$ is obtained by multiplying both sides of Eq. (20.31) by e^{j2t} and taking the real part. Thus:

The real sinusoidal steady-state response

$$\begin{aligned} i(t) &= I_m \cos(\omega t + \phi) \\ &= 3\sqrt{2}\cos\left(2t - \frac{\pi}{4}\right) \end{aligned} \tag{20.35}$$

This agrees with the response derived before.

Although the analysis was straightforward, we have not yet taken advantage of the full power of the complex representation. At the moment, we still have to determine the appropriate differential equation. By introducing the concept of the *phasor*, however, we are able to eliminate this requirement, and the result is the simplification of sinusoidal analysis.

20.3 The Phasor

A sinusoidal voltage or current *at a given frequency* is characterized by only two parameters, an amplitude and a phase. The complex representation of the voltage or current is characterized by a magnitude and an angle. For example, the assumed sinusoidal form of the current response in the previous example was:

$$I_m \cos(\omega t + \phi) \tag{20.36}$$

and the corresponding representation of this current in complex form is:

$$I_m e^{j(\omega t + \phi)} \tag{20.37}$$

Once I_m and ϕ are specified, the current is exactly defined. Throughout any linear circuit operating in the sinusoidal steady-state at a single frequency ω, every voltage and current may be characterized completely by a knowledge of its amplitude and phase angle.

A sinusoid of a given frequency is specified by an amplitude and phase

All sinusoidal responses in a linear circuit have a frequency of ω. Therefore, instead of writing $I_m \cos(\omega t + \phi)$, we could just say "amplitude I_m" and "phase ϕ".

A complex response of a given frequency is specified by a magnitude and angle

All complex responses in a linear circuit have the factor $e^{j\omega t}$. Therefore, instead of writing $I_m e^{j(\omega t + \phi)}$, we could just say "magnitude I_m" and "angle ϕ".

Thus, we can simplify the voltage source and current response of the example by representing them concisely as complex numbers:

$$V_m e^{j0^\circ} \quad \text{and} \quad I_m e^{j\phi} \tag{20.38}$$

We usually write the complex representation in polar form. Thus, the source voltage:

$$v(t) = V_m \cos(\omega t) \tag{20.39}$$

is represented in complex form as:

$$\mathbf{V} = V_m \angle 0° \tag{20.40}$$

and the current response:

A general sinusoid…

$$i(t) = I_m \cos(\omega t + \phi) \tag{20.41}$$

as:

…and its phasor representation

$$\mathbf{I} = I_m \angle \phi \tag{20.42}$$

The abbreviated complex representation is called a *phasor*. Phasors are printed in boldface because they are effectively like a *vector*, they have a magnitude and direction (angle). In hand writing, we normally place a tilde underneath:

$$\underset{\sim}{V} = V_m \angle 0° \quad \text{and} \quad \underset{\sim}{I} = I_m \angle \phi \tag{20.43}$$

Capital letters are used to represent phasors because they are constants – they are not functions of time.

Introducing the "frequency-domain"

In general, we refer to $x(t)$ as a *time-domain representation* and the corresponding phasor $\mathbf{X}$ as a *frequency-domain representation*.

We can see that the *magnitude* of the complex representation is the *amplitude* of the sinusoid and the *angle* of the complex representation is the *phase* of the sinusoid.

It is a simple matter to convert a signal from the time-domain to the frequency-domain – it is achieved by inspection:

The time-domain and frequency-domain relationships for a sinusoid

$$x(t) = A\cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{X} = Ae^{j\phi}$$

$$\text{amplitude} \quad \Leftrightarrow \quad \text{magnitude} \tag{20.44}$$

$$\text{phase} \quad \Leftrightarrow \quad \text{angle}$$

EXAMPLE 20.1 Phasor Representation

If $x(t) = 3\sin(\omega t - 30°)$ then we have to convert to our cos notation: $x(t) = 3\cos(\omega t - 120°)$. Therefore $\mathbf{X} = 3\angle -120°$.

Note carefully that $\mathbf{X} \neq 3\cos(\omega t - 120°)$. All we can say is that $x(t) = 3\cos(\omega t - 120°)$ is *represented* by $\mathbf{X} = 3\angle -120°$.

The convenience of complex numbers extends beyond their compact representation of the amplitude and phase. The sum of two phasors corresponds to the sinusoid which is the sum of the two component sinusoids represented by the phasors. That is, if $x_3(t) = x_1(t) + x_2(t)$ where $x_1(t)$, $x_2(t)$ and $x_3(t)$ are sinusoids with the same frequency, then $\mathbf{X}_3 = \mathbf{X}_1 + \mathbf{X}_2$.

Phasors make manipulating sinusoids of the same frequency easy

EXAMPLE 20.2 Phasor Representation

If $x_3(t) = \cos\omega t - 2\sin\omega t$ then $\mathbf{X}_3 = 1\angle 0° - 2\angle -90° = 1 + j2 = 2.24\angle 63°$ which corresponds to $x_3(t) = 2.24\cos(\omega t + 63°)$.

20.3.1 Formalisation of the Relationship between Phasor and Sinusoid

Using Euler's identity:

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{20.45}$$

we have:

$$Ae^{j\phi}e^{j\omega t} = Ae^{j(\omega t+\phi)} = A\cos(\omega t+\phi) + j\sin(\omega t+\phi) \tag{20.46}$$

We can see that the sinusoid $A\cos(\omega t+\phi)$ represented by the phasor $\mathbf{X} = Ae^{j\phi}$ is equal to the real part of $\mathbf{X}e^{j\omega t}$. Therefore:

The phasor / time-domain relationship

$$x(t) = \mathrm{Re}\left\{\mathbf{X}e^{j\omega t}\right\} \tag{20.47}$$

This can be visualised as:

Graphical interpretation of rotating phasor / time-domain relationship

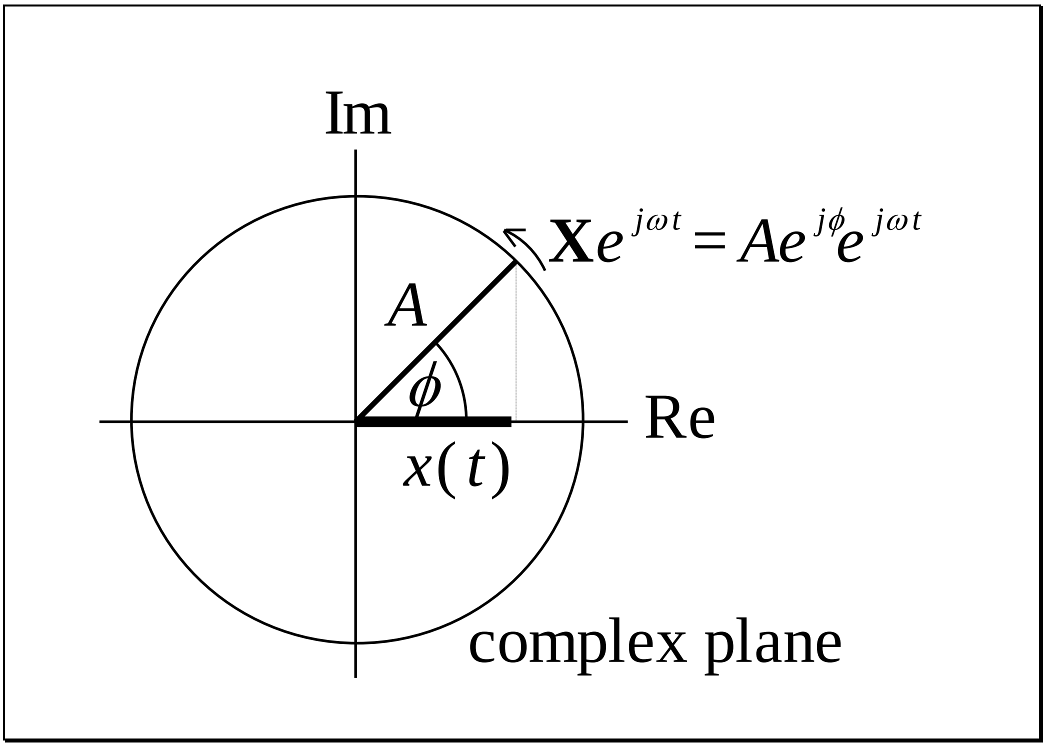


Figure 20.7 – Graphical interpretation of a phasor

Run the Phasor simulation program to see this view of phasors in action!

20.3.2 Graphical Illustration of the Relationship between a Phasor and its Corresponding Sinusoid

Consider the representation of a sinusoid by its phasor: $x(t) = \mathrm{Re}\{\mathbf{X}e^{j\omega t}\}$. Graphically, $x(t)$ can be "generated" by taking the projection of the rotating phasor formed by multiplying $\mathbf{X}$ by $e^{j\omega t}$, onto the real axis:

A sinusoid can be generated by taking the real part of a rotating complex number

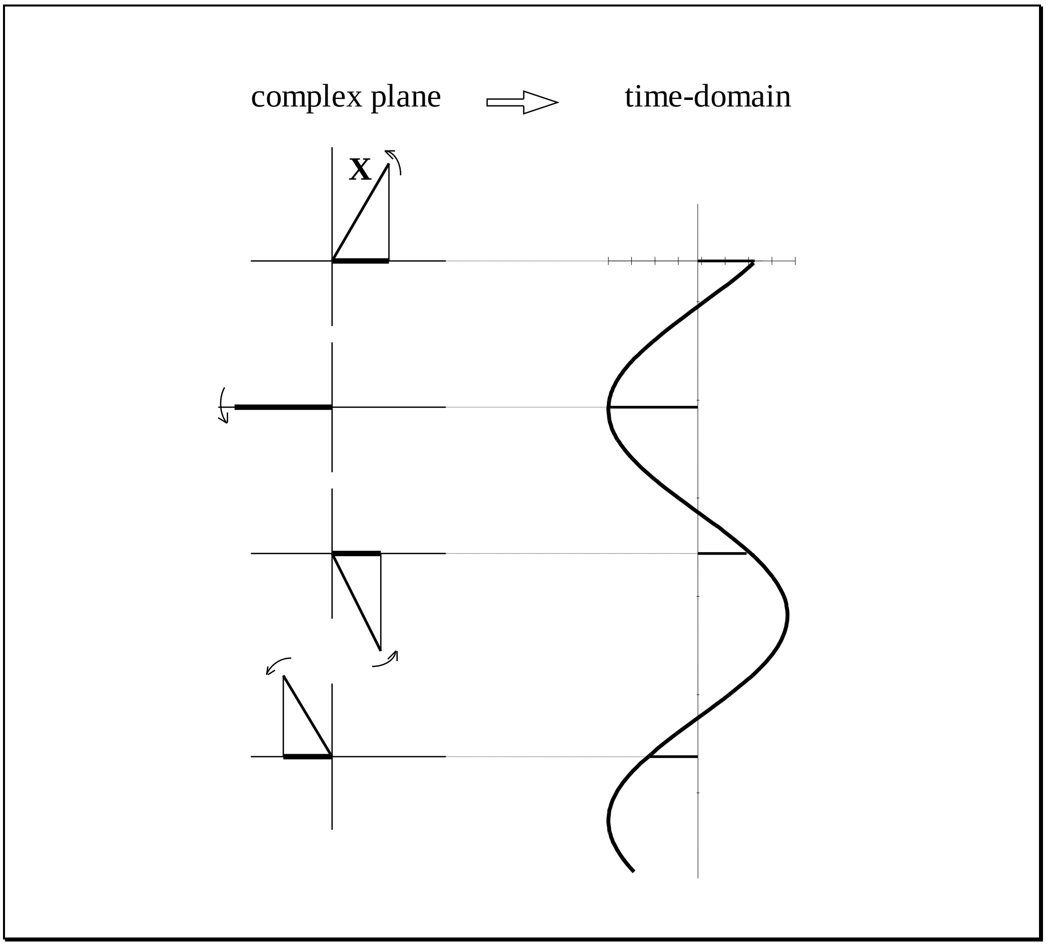


Figure 20.8 – Graphical illustration of a phasor generating a sinusoid

Phasor Representations

Phasors can be represented in four different ways:

$\mathbf{X} = X_m \angle \phi$	polar form
$\mathbf{X} = X_m e^{j\phi}$	exponential form
$\mathbf{X} = X_m(\cos\phi + j\sin\phi)$	trigonometric form
$\mathbf{X} = a + jb$	rectangular form

20.4 Phasor Relationships for *R*, *L* and *C*

Phasor relationships for the passive elements

Now that we can transform into and out of the frequency-domain, we can derive the phasor relationships for each of the three passive circuit elements. This will lead to a great simplification of sinusoidal steady-state analysis.

We will begin with the defining time-domain equation for each of the elements, and then let both the voltage and current become complex quantities. After dividing throughout the equation by $e^{j\omega t}$, the desired relationship between the phasor voltage and phasor current will become apparent.

20.4.1 Phasor Relationships for a Resistor

The resistor provides the simplest case. The defining time-domain equation is:

$$v(t) = Ri(t) \tag{20.48}$$

If we apply a complex voltage $V_m e^{j(\omega t+\theta)}$ and assume a complex current $I_m e^{j(\omega t+\phi)}$, we obtain:

$$V_m e^{j(\omega t+\theta)} = RI_m e^{j(\omega t+\phi)} \tag{20.49}$$

By dividing throughout by $e^{j\omega t}$, we find:

$$V_m e^{j\theta} = RI_m e^{j\phi} \tag{20.50}$$

or in polar form:

$$V_m \angle\theta = RI_m \angle\phi \tag{20.51}$$

But $V_m\angle\theta$ and $I_m\angle\phi$ are just the voltage and current phasors $\mathbf{V}$ and $\mathbf{I}$. Thus:

Phasor **V-I** relationship for a resistor

$$\boxed{\mathbf{V} = R\mathbf{I}} \tag{20.52}$$

Equality of the angles θ and ϕ is apparent, and the current and voltage are thus in phase.

The voltage-current relationship in phasor form for a resistor has the same form as the relationship between the time-domain voltage and current as illustrated below:

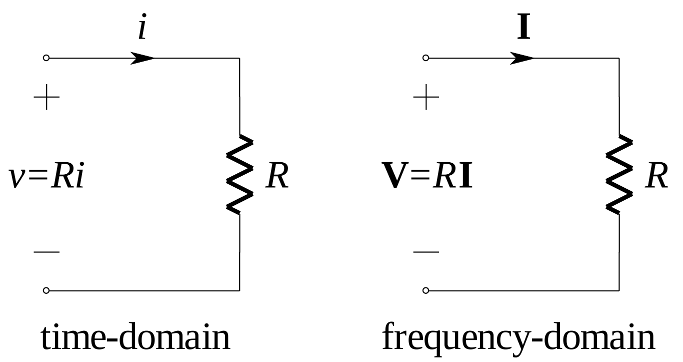


Figure 20.9 – Resistor relationships in the time- and frequency-domain

EXAMPLE 20.3 Phasor Analysis with a Resistor

Assume a voltage of $8\cos(100t - 50°)$ across a $4\,\Omega$ resistor. Working in the time-domain, the current is:

$$i(t) = \frac{v(t)}{R} = 2\cos(100t - 50°)$$

The phasor form of the same voltage is $8\angle -50°$, and therefore:

$$\mathbf{I} = \frac{\mathbf{V}}{R} = 2\angle -50°$$

If we transform back to the time-domain, we get the same expression for the current.

No work is saved for a resistor by analysing in the frequency-domain – because the resistor has a linear relationship between voltage and current.

20.4.2 Phasor Relationships for an Inductor

The defining time-domain equation is:

$$v(t) = L\frac{di(t)}{dt} \tag{20.53}$$

After applying the complex voltage and current equations, we obtain:

$$V_m e^{j(\omega t+\theta)} = L\frac{d}{dt}\left(I_m e^{j(\omega t+\phi)}\right) \tag{20.54}$$

Taking the indicated derivative:

$$V_m e^{j(\omega t+\theta)} = j\omega L I_m e^{j(\omega t+\phi)} \tag{20.55}$$

By dividing throughout by $e^{j\omega t}$, we find:

$$V_m e^{j\theta} = j\omega L I_m e^{j\phi} \tag{20.56}$$

Thus the desired phasor relationship is:

Phasor **V-I** relationship for an inductor

$$\mathbf{V} = j\omega L\mathbf{I} \tag{20.57}$$

The time-domain equation Eq. (20.53) has become an algebraic equation in the frequency-domain. The angle of $j\omega L$ is exactly $+90°$ and you can see from Eq. (20.56) that $\theta = 90° + \phi$. $\mathbf{I}$ must therefore *lag* $\mathbf{V}$ by $90°$ in an inductor.

The phasor relationship for an inductor is indicated below:

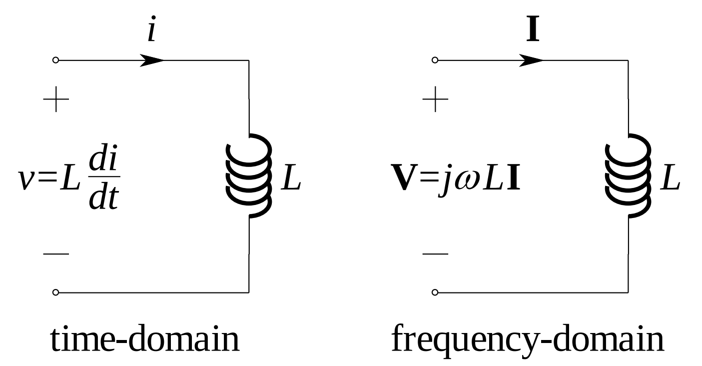


Figure 20.10 – Inductor relationships in the time- and frequency-domain

EXAMPLE 20.4 Phasor Analysis with an Inductor

Assume a voltage of $8\cos(100t-50°)$ across a $4\,\mathrm{H}$ inductor. Working in the time-domain, the current is:

$$\begin{aligned} i(t) &= \int \frac{v(t)}{L}dt \\ &= \int 2\cos(100t-50°)dt \\ &= 0.02\sin(100t-50°) \\ &= 0.02\cos(100t-140°) \end{aligned}$$

The phasor form of the same voltage is $8\angle-50°$, and therefore:

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L} = \frac{8\angle-50°}{100(4)\angle 90°} = 0.02\angle-140°$$

If we transform back to the time-domain, we get the same expression for the current.

20.4.3 Phasor Relationships for a Capacitor

The defining time-domain equation is:

$$i(t) = C\frac{dv(t)}{dt} \tag{20.58}$$

After applying the complex voltage and current equations, we obtain:

$$I_m e^{j(\omega t+\phi)} = C\frac{d}{dt}\left(V_m e^{j(\omega t+\theta)}\right) \tag{20.59}$$

Taking the indicated derivative:

$$I_m e^{j(\omega t+\phi)} = j\omega C V_m e^{j(\omega t+\theta)} \tag{20.60}$$

By dividing throughout by $e^{j\omega t}$, we find:

$$I_m e^{j\phi} = j\omega C V_m e^{j\theta} \tag{20.61}$$

Thus the desired phasor relationship is:

Phasor **V-I** relationship for a capacitor

$$\mathbf{I} = j\omega C\mathbf{V} \tag{20.62}$$

Thus $\mathbf{I}$ *leads* $\mathbf{V}$ by $90°$ in a capacitor.

The time-domain and frequency-domain representations are compared below:

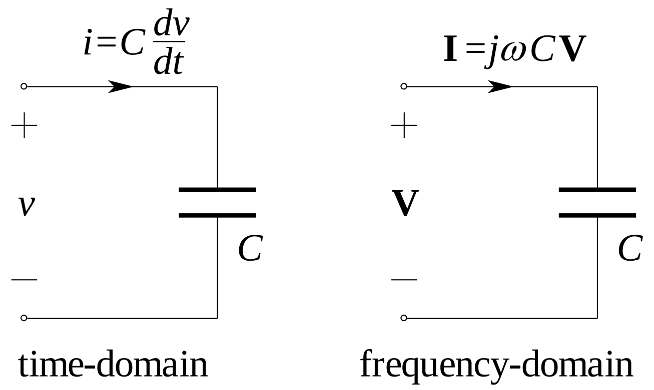


Figure 20.11 – Capacitor relationships in the time- and frequency-domain

EXAMPLE 20.5 Phasor Analysis with a Capacitor

Assume a voltage of $8\cos(100t - 50°)$ across a $4\,\mathrm{F}$ capacitor. Working in the time-domain, the current is:

$$\begin{aligned} i(t) &= C\frac{dv(t)}{dt} \\ &= 4\frac{d}{dt}8\cos(100t - 50°) \\ &= -3200\sin(100t - 50°) \\ &= 3200\cos(100t + 40°) \end{aligned}$$

The phasor form of the same voltage is $8\angle -50°$, and therefore:

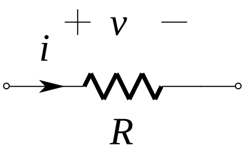
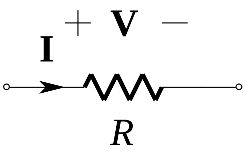
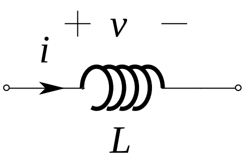
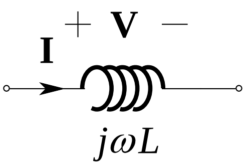
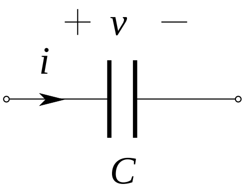
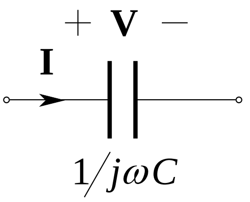
$$\mathbf{I} = j\omega C\mathbf{V} = 100(4)\angle 90° \times 8\angle -50° = 3200\angle 40°$$

If we transform back to the time-domain, we get the same expression for the current.

20.4.4 Summary of Phasor Relationships for *R*, *L* and *C*

We have now obtained the phasor $\mathbf{V}-\mathbf{I}$ relationships for the three passive elements. These results are summarized in the table below:

Summary of phasor **V-I** relationships for the passive elements

Time-domain		*Frequency-domain*	
	$v = Ri$	$\mathbf{V} = R\mathbf{I}$	
	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$	
	$v = \dfrac{1}{C}\int i\,dt$	$\mathbf{V} = \dfrac{1}{j\omega C}\mathbf{I}$	

All the phasor equations are algebraic. Each is also linear, and the equations relating to inductance and capacitance bear a great similarity to Ohm's Law.

Before we embark on using the phasor relationships in circuit analysis, we need to verify that KVL and KCL work for phasors. KVL in the time-domain is:

$$v_1(t)+v_2(t)+\cdots+v_n(t)=0 \tag{20.63}$$

If all voltages are sinusoidal, we can now use Euler's identity to replace each real sinusoidal voltage by the complex voltage having the same real part, divide by $e^{j\omega t}$ throughout, and obtain:

KVL and KCL are obeyed by phasors

$$\mathbf{V}_1+\mathbf{V}_2+\cdots+\mathbf{V}_n=0 \tag{20.64}$$

Thus KVL holds. KCL also holds by a similar argument.

20.4.5 Analysis Using Phasor Relationships

We now return to the series *RL* circuit that we considered several times before, shown as (a) in the figure below. We draw the circuit in the frequency-domain, as shown in (b):

A circuit and its frequency-domain equivalent

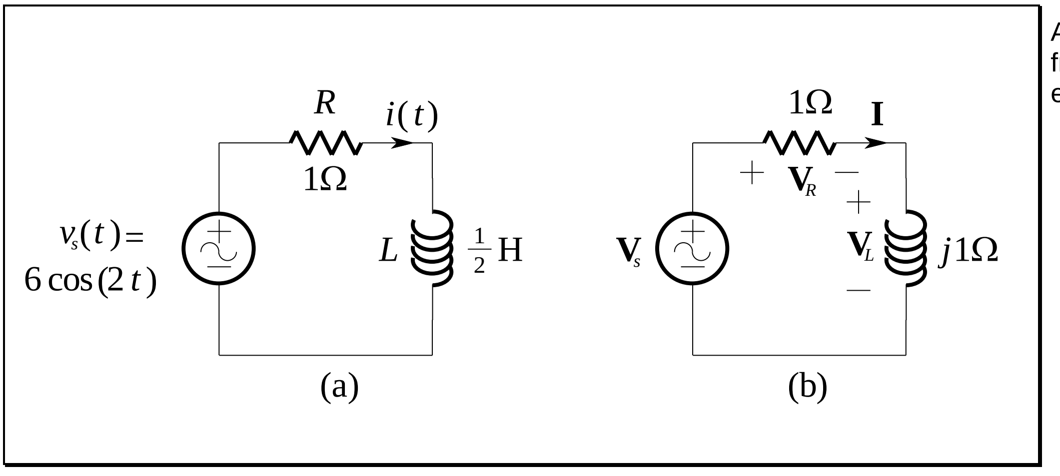


Figure 20.12 – Analysis of the *RL* circuit in the frequency-domain

Note that all passive elements in the frequency-domain have the units of *ohms*. From KVL in the frequency-domain:

$$\mathbf{V}_R + \mathbf{V}_L = \mathbf{V}_s \tag{20.65}$$

We now insert the recently obtained $\mathbf{V}-\mathbf{I}$ relationships for the elements:

$$\begin{aligned} R\mathbf{I} + j\omega L\mathbf{I} &= \mathbf{V}_s \\ \mathbf{I} + j\mathbf{I} &= \mathbf{V}_s \end{aligned} \tag{20.66}$$

The phasor current is then found:

$$\mathbf{I} = \frac{\mathbf{V}_s}{1+j} \tag{20.67}$$

The source has a magnitude of 6 and a phase of $0°$ (it is the reference by which all other phase angles are measured). Thus:

The response of the circuit in the frequency-domain

$$\mathbf{I} = \frac{6\angle 0°}{1+j} \tag{20.68}$$

The current may be transformed to the time-domain by first writing it in polar form:

$$\begin{aligned}\mathbf{I} &= 3\sqrt{2}\angle -\pi/4 \\ &= I_m \angle \phi\end{aligned} \tag{20.69}$$

Transforming back to the time-domain we get:

The response of the circuit in the time-domain

$$\begin{aligned}i(t) &= I_m \cos(2t+\phi) \\ &= 3\sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)\end{aligned} \tag{20.70}$$

which is the same result as we obtained before the “hard way”.

20.5 Impedance

The voltage-current relationships for the three passive elements in the frequency-domain are:

$$\mathbf{V} = R\mathbf{I} \qquad \mathbf{V} = j\omega L\mathbf{I} \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C} \tag{20.71}$$

If these equations are written as phasor-voltage phasor-current ratios, we get:

$$\frac{\mathbf{V}}{\mathbf{I}} = R \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C} \tag{20.72}$$

Phasor **V-I** relationships for the passive elements

These ratios are simple functions of the element values, and in the case of the inductor and capacitor, frequency. We treat these ratios in the same manner we treat resistances, with the exception that they are complex quantities and all algebraic manipulations must be those appropriate for complex numbers.

We define the ratio of the phasor voltage to the phasor current as *impedance*, symbolized by the letter $\mathbf{Z}$:

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \tag{20.73}$$

Impedance defined

The impedance is a complex quantity having the dimensions of ohms.

Impedance is *not* a phasor and cannot be transformed to the time-domain by multiplying by $e^{j\omega t}$ and taking the real part.

In the table below, we show how we can represent a resistor, inductor or capacitor in the time-domain with its frequency-domain impedance:

Impedances of the three passive elements

Time-domain	*Frequency-domain*
R	R
L	$j\omega L$
C	$1/j\omega C$

Impedances may be combined in series and parallel by the same rules we use for resistances.

In a circuit diagram, a general impedance is represented by a rectangle:

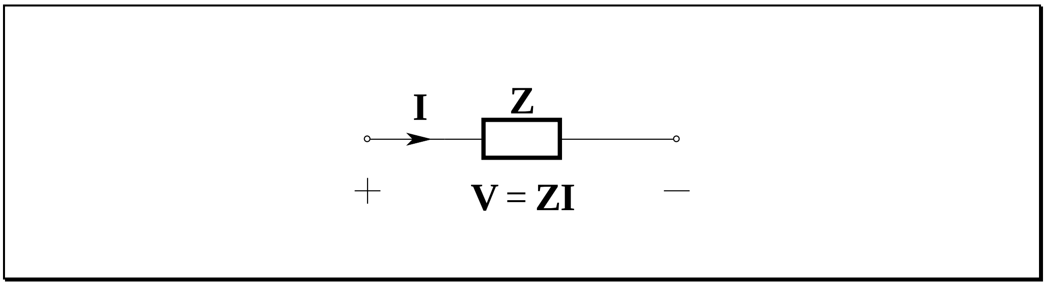
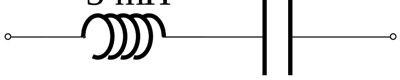


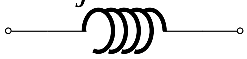
Figure 11.1

EXAMPLE 20.6 Impedance of an Inductor and Capacitor in Series

We have an inductor and capacitor in series:



At $\omega = 10^4 \text{ rads}^{-1}$, the impedance of the inductor is $\mathbf{Z}_L = j\omega L = j50\,\Omega$ and the impedance of the capacitor is $\mathbf{Z}_C = 1/j\omega C = -j1\,\Omega$. Thus the series combination is equivalent to $\mathbf{Z}_{eq} = \mathbf{Z}_L + \mathbf{Z}_C = j50 - j1 = j49\,\Omega$:



The impedance of inductors and capacitors is a function of frequency, and this equivalent impedance is only valid at $\omega = 10^4 \text{ rads}^{-1}$. For example, if $\omega = 5000 \text{ rads}^{-1}$, then the impedance would be $\mathbf{Z}_{eq} = j23\,\Omega$.

Impedance may be expressed in either polar or rectangular form.

In polar form an impedance is represented by:

$$\mathbf{Z} = |\mathbf{Z}|\angle\theta \tag{20.74}$$

No special names or symbols are assigned to the magnitude and angle. For example, an impedance of $100\angle -60°\,\Omega$ is described as having an impedance magnitude of $100\,\Omega$ and an angle of $-60°$.

Impedance is composed of a resistance (real part) and a reactance (imaginary part)

In rectangular form an impedance is represented by:

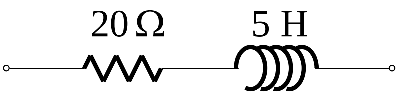
$$\mathbf{Z} = R + jX \tag{20.75}$$

The real part, R, is termed the *resistive component*, or *resistance*. The imaginary component, X, including sign, but excluding j, is termed the *reactive component*, or *reactance*. The impedance $100\angle -60°\ \Omega$ in rectangular form is $50 - j86.6\ \Omega$. Thus, its resistance is $50\ \Omega$ and its reactance is $-86.6\ \Omega$.

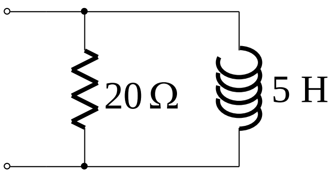
It is important to note that the resistive component of the impedance is not necessarily equal to the resistance of the resistor which is present in the circuit.

EXAMPLE 20.7 Impedance of a Resistor and Inductor in Series

Consider a resistor and an inductor in series:



At $\omega = 4\ \text{rads}^{-1}$, the equivalent impedance is $\mathbf{Z}_{eq} = 20 + j20\ \Omega$. In this case the resistive component of the impedance is equal to the resistance of the resistor because the network is a simple series network. Now consider the same elements placed in parallel:



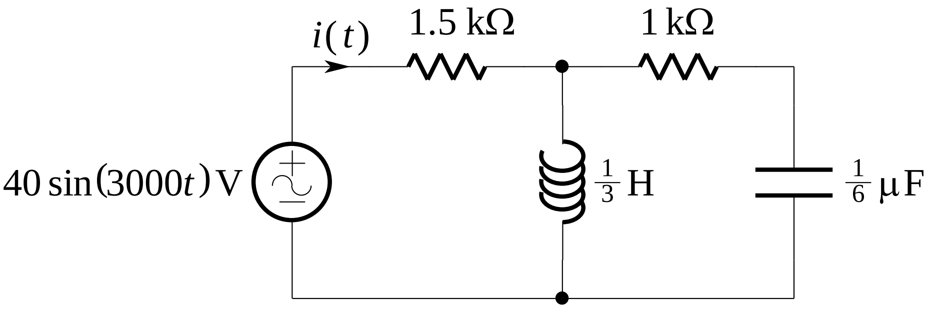
The equivalent impedance is:

$$\mathbf{Z}_{eq} = \frac{20(j20)}{20 + j20} = 10 + \text{j}10\ \Omega$$

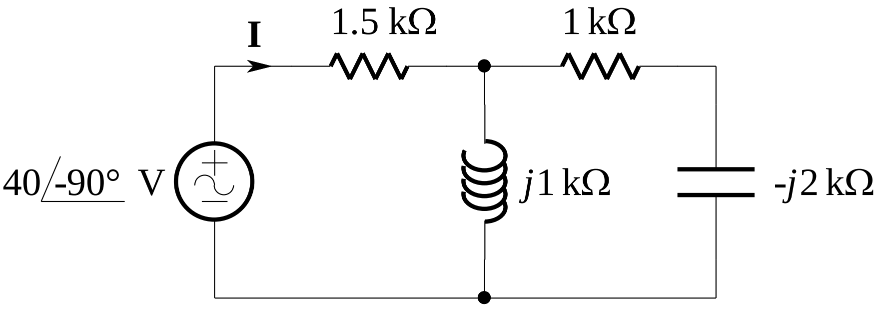
The resistive component of the impedance is now $10\ \Omega$.

EXAMPLE 20.8 Circuit Analysis using Impedance

We will use the impedance concept to analyse the *RLC* circuit shown below:



The circuit is shown in the time-domain, and a time-domain response is required. However, analysis should be carried out in the frequency-domain. We therefore begin by drawing the frequency-domain circuit – the source is transformed to the frequency-domain, becoming $40\angle -90°$, the response is transformed to the frequency-domain, being represented as $\mathbf{I}$, and the impedances of the inductor and capacitor, determined at $\omega = 3000$ rads^{-1}, are $j1\,\text{k}\Omega$ and $-j2\,\text{k}\Omega$ respectively. The frequency-domain circuit is shown below:



The equivalent impedance offered to the source is:

$$\begin{aligned}\mathbf{Z}_{eq} &= 1.5 + \frac{(j1)(1-j2)}{j1+1-j2} = 1.5 + \frac{2+j}{1-j} \\ &= 1.5 + \frac{2+j}{1-j}\frac{1+j}{1+j} = 1.5 + \frac{1+j3}{2} \\ &= 2 + j1.5 = 2.5\angle 36.9° \quad \text{k}\Omega\end{aligned}$$

The phasor current is thus:

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}_{eq}} = \frac{40\angle -90^\circ}{2.5\angle 36.9^\circ} = 16\angle -126.9^\circ \quad \text{mA}$$

Upon transforming the current to the time-domain, the desired response is obtained:

$$i(t) = 16\cos(3000t - 126.9^\circ) \quad \text{mA}$$

20.6 Admittance

The reciprocal of impedance can offer some convenience in the sinusoidal steady-state analysis of circuits. We define *admittance* as the ratio of phasor current to phasor voltage:

Admittance defined

$$\mathbf{Y} = \frac{\mathbf{I}}{\mathbf{V}} \tag{20.76}$$

and thus:

Admittance is the reciprocal of impedance

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} \tag{20.77}$$

The real part of the admittance is the *conductance* G, and the imaginary part of the admittance is the *susceptance* B. Thus:

Admittance is composed of a conductance (real part) and a susceptance (imaginary part)

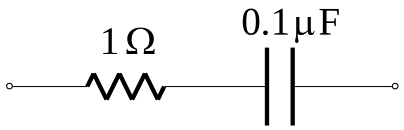
$$\mathbf{Y} = G + jB = \frac{1}{\mathbf{Z}} = \frac{1}{R + jX} \tag{20.78}$$

This formula should be scrutinized carefully. It does *not* mean that $G = 1/R$ (unless $\mathbf{Z} = R$, a pure resistance), nor does it mean $B = 1/X$.

Admittance, conductance and susceptance are all measured in siemens (S).

EXAMPLE 20.9 Admittance of a Resistor and Capacitor in Series

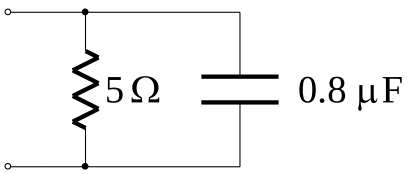
Consider a resistor and a capacitor in series:



At $\omega = 5\ \text{Mrads}^{-1}$, the equivalent impedance is $\mathbf{Z}_{eq} = 1 - j2\ \Omega$. Its admittance is:

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{1}{1-j2} = \frac{1}{1-j2}\frac{1+j2}{1+j2}$$
$$= 0.2 + j0.4\ \text{S}$$

It should be apparent that the equivalent admittance of a circuit consisting of a number of parallel branches is the sum of the admittances of the individual branches. Thus, the admittance obtained above is equivalent to:



only at $\omega = 5\ \text{Mrads}^{-1}$. As a check , the equivalent impedance of the parallel network at $\omega = 5\ \text{Mrads}^{-1}$ is:

$$\mathbf{Z}_{eq} = \frac{5(-j2.5)}{5-j2.5} = 1 - j2\ \Omega$$

as before.

20.7 Summary

- Sinusoids are important theoretically and practically. A sinusoidal source yields a sinusoidal response.

- The sinusoidal forced response is also known as the sinusoidal steady-state – the condition which is reached after the transient response has died out.

- A complex forcing function produces a complex response – the real part of the forcing function creates the real part of the response.

- The application of a complex forcing function to a linear circuit turns the describing differential equation into a complex algebraic equation.

- The phasor representation of a sinusoid captures the amplitude and phase information in a complex number – amplitude corresponds to magnitude, and phase corresponds to angle.

Time-Domain	**Frequency-Domain**
sinusoid	phasor
$x(t) = A\cos(\omega t + \phi)$	$\mathbf{X} = Ae^{j\phi}$
amplitude	magnitude
phase	angle

- Phasor $\mathbf{V} - \mathbf{I}$ relationships for the three passive elements lead to the concept of frequency-domain impedance. The impedances of the three passive elements are: $\mathbf{Z}_R = R$, $\mathbf{Z}_L = j\omega L$, $\mathbf{Z}_C = 1/j\omega C$. Impedances can be combined and manipulated like resistors except we use complex algebra.

- Impedance consists of a real resistive component and an imaginary reactive component: $\mathbf{Z} = R + jX$.

- Admittance is defined as the inverse of impedance: $\mathbf{Y} = 1/\mathbf{Z} = G + jB$.

20.8 References

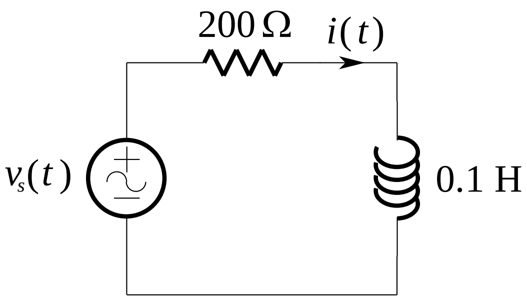
Bedient, P. & Rainville, E.: *Elementary Differential Equations*, 6th *Ed*. Macmillan Publishing Co., 1981.

Hayt, W. & Kemmerly, J.: *Engineering Circuit Analysis*, 3rd *Ed*., McGraw-Hill, 1984.

Exercises

1.

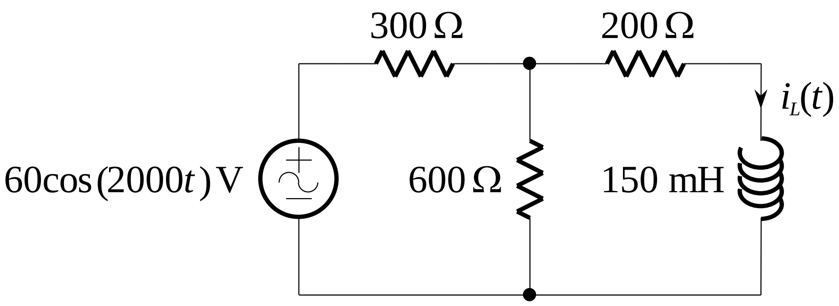
Consider the circuit shown below:



Let $i = 2\sin(500t - 40°)$ A. Determine the source voltage $v_s(t)$.

2.

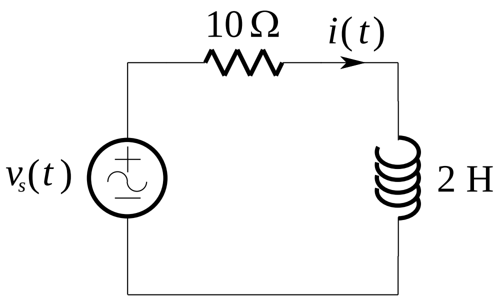
Consider the circuit shown below:



Find $i_L(t)$.

3.

Consider the circuit shown below:



Let $v_s = 20\cos 5t + 30\cos 10t$ V. Find $i(t)$.

4.

Give in polar form:

(a) $20\angle 110° - 8\angle 40°$ (b) $(6.1 - j3.82)/(1.17 + j0.541)$

Give in rectangular form:

(c) $j/(6.3 - j9.71)$ (d) $e^{j2.1\angle 52°}$

5.

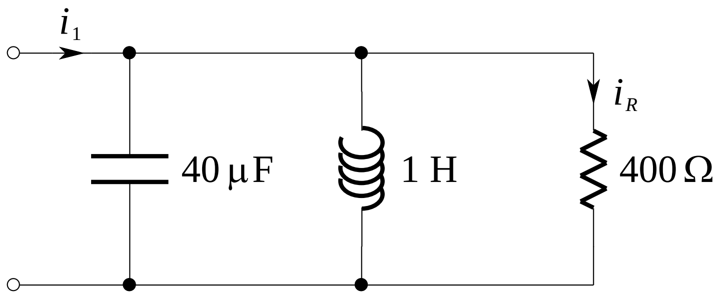
Assume that only three currents, i_1, i_2 and i_3, enter a certain node.

(a) Find $i_1(t)$ if $\mathbf{I}_2 = 65\angle -110°$ mA and $\mathbf{I}_3 = 45\angle -50°$ mA

(b) Find $\mathbf{I}_2$ if $i_1(t) = 55\cos(400t + 40°)$ mA and $i_3(t) = 35\sin(400t - 70°)$ mA.

6.

The circuit below is operated at $\omega = 100 \text{ rads}^{-1}$.



(a) If $\mathbf{I}_R = 0.01\angle 20° \text{ A}$, find $\mathbf{I}_1$.

(b) If $\mathbf{I}_1 = 2\angle -30° \text{ A}$, find $\mathbf{I}_R$.

7.

Using a 1 H inductor and a 1 μF capacitor, at what frequency (in hertz) may an impedance be obtained having a magnitude of 2000 Ω if the two elements are combined in:

(a) series (b) parallel

8.

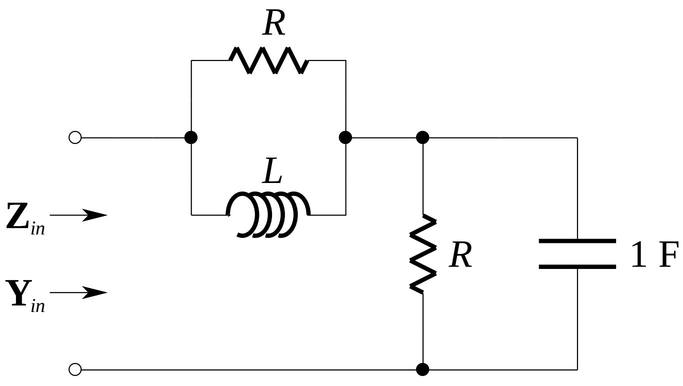
A 10 μF capacitor and a 25 Ω resistor are in parallel. What size inductor should be placed in series with this parallel combination so that the impedance of the final series network has zero reactance at 8 krads^{-1}?

9.

What size capacitor should be placed in series with the series combination of 800 Ω and 20 mH to give an admittance whose magnitude is 1 mS at $\omega = 10 \text{ krads}^{-1}$?

10.

If the input admittance and impedance of the network shown below are equal at every frequency, find *R* and *L*.



Joseph Fourier (1768-1830) *(Jo´ sef Foor´ yay)*

Fourier is famous for his study of the flow of heat in metallic plates and rods. The theory that he developed now has applications in industry and in the study of the temperature of the Earth's interior. He is also famous for the discovery that many functions could be expressed as infinite sums of sine and cosine terms, now called a trigonometric series, or Fourier series.

Fourier first showed talent in literature, but by the age of thirteen, mathematics became his real interest. By fourteen, he had completed a study of six volumes of a course on mathematics. Fourier studied for the priesthood but did not end up taking his vows. Instead he became a teacher of mathematics. In 1793 he became involved in politics and joined the local Revolutionary Committee. As he wrote:-

> *As the natural ideas of equality developed it was possible to conceive the sublime hope of establishing among us a free government exempt from kings and priests, and to free from this double yoke the long-usurped soil of Europe. I readily became enamoured of this cause, in my opinion the greatest and most beautiful which any nation has ever undertaken.*

Fourier became entangled in the French Revolution, and in 1794 he was arrested and imprisoned. He feared he would go to the guillotine but political changes allowed him to be freed. In 1795, he attended the Ecole Normal and was taught by, among others, Lagrange and Laplace. He started teaching again, and began further mathematical research. In 1797, after another brief period in prison, he succeeded Lagrange in being appointed to the chair of analysis and mechanics. He was renowned as an outstanding lecturer but did not undertake original research at this time.

In 1798 Fourier joined Napoleon on his invasion of Egypt as scientific adviser. The expedition was a great success (from the French point of view) until August 1798 when Nelson's fleet completely destroyed the French fleet in the Battle of the Nile, so that Napoleon found himself confined to the land he was occupying. Fourier acted as an administrator as French type political institutions and administrations were set up. In particular he helped establish educational facilities in Egypt and carried out archaeological explorations.

While in Cairo, Fourier helped found the Institute d'Égypte and was put in charge of collating the scientific and literary discoveries made during the time in Egypt. Napoleon abandoned his army and returned to Paris in 1799 and soon held absolute power in France. Fourier returned to France in 1801 with the remains of the expeditionary force and resumed his post as Professor of Analysis at the Ecole Polytechnique.

The Institute d'Égypte was responsible for the completely serendipitous discovery of the Rosetta Stone in 1799. The three inscriptions on this stone in two languages and three scripts (hieroglyphic, demotic and Greek) enabled Thomas Young and Jean-François Champollion, a protégé of Fourier, to invent a method of translating hieroglyphic writings of ancient Egypt in 1822.

Napoleon appointed Fourier to be Prefect at Grenoble where his duties were many and varied – they included draining swamps and building highways. It was during his time in Grenoble that Fourier did his important mathematical work on the theory of heat. His work on the topic began around 1804 and by 1807 he had completed his important memoir *On the Propagation of Heat in Solid Bodies*. It caused controversy – both Lagrange and Laplace objected to Fourier's expansion of functions as trigonometric series.

> *…it was in attempting to verify a third theorem that I employed the procedure which consists of multiplying by* $\cos x\, dx$ *the two sides of the equation*
>
> $$\phi(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \ldots$$
>
> *and integrating between* $x = 0$ *and* $x = \pi$. *I am sorry not to have known the name of the mathematician who first made use of this method because I would have cited him. Regarding the researches of d'Alembert and Euler could one not add that if they knew this expansion they made but a very imperfect use of it. They were both persuaded that an arbitrary…function could never be resolved in a series of this kind, and it does not seem that any one had developed a constant in cosines of multiple arcs* [i.e. found a_1, a_2,…, with $1 = a_1 \cos x + a_2 \cos 2x + \ldots$ for $-\pi/2 < x < \pi/2$] *the first problem which I had to solve in the theory of heat.*

This extract is from a letter found among Fourier's papers, and unfortunately lacks the name of the addressee, but was probably intended for Lagrange.

Other people before Fourier had used expansions of the form $f(x) \sim \sum_{r=-\infty}^{\infty} a_r \exp(irt)$ but Fourier's work extended this idea in two totally new ways. One was the "Fourier integral" (the formula for the Fourier series coefficients) and the other marked the birth of Sturm-Liouville theory (Sturm and Liouville were nineteenth century mathematicians who found solutions to many classes of partial differential equations arising in physics that were analogous to Fourier series).

Napoleon was defeated in 1815 and Fourier returned to Paris. Fourier was elected to the Académie des Sciences in 1817 and became Secretary in 1822. Shortly after, the Academy published his prize winning essay *Théorie analytique de la chaleur* (*Analytical Theory of Heat*). In this he obtains for the first time the equation of heat conduction, which is a partial differential equation in three dimensions. As an application he considered the temperature of the ground at a certain depth due to the sun's heating. The solution consists of a yearly component and a daily component. Both effects die off exponentially with depth but the high frequency daily effect dies off much more rapidly than the low frequency yearly effect. There is also a phase lag for the daily and yearly effects so that at certain depths the temperature will be completely out of step with the surface temperature.

All these predictions are confirmed by measurements which show that annual variations in temperature are imperceptible at quite small depths (this accounts for the permafrost, i.e. permanently frozen subsoil, at high latitudes) and that daily variations are imperceptible at depths measured in tenths of metres. A reasonable value of soil thermal conductivity leads to a prediction that annual temperature changes will lag by six months at about 2–3 metres depth. Again this is confirmed by observation and, as Fourier remarked, gives a good depth for the construction of cellars.

As Fourier grew older, he developed at least one peculiar notion. Whether influenced by his stay in the heat of Egypt or by his own studies of the flow of heat in metals, he became obsessed with the idea that extreme heat was the natural condition for the human body. He was always bundled in woollen clothing, and kept his rooms at high temperatures. He died in his sixty-third year, "thoroughly cooked".

References

Körner, T.W.: *Fourier Analysis*, Cambridge University Press, 1988.