22 AC Power

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Introduction

In previous analyses our major concern has been with determining voltages and currents. In many applications (e.g. electric utilities), energy or power supplied and absorbed are extremely important parameters. Knowing the instantaneous power is useful, since many electrical and electronic devices have maximum instantaneous or "peak" power ratings that, for satisfactory operation, should not be exceeded. By averaging instantaneous power we get *average power*, which is the average rate at which energy is supplied or absorbed. Power usages vary from a fraction of a watt for small electronic circuits to millions of watts for large electric utilities.

We have previously defined power to be the product of voltage and current. This is only true for instantaneous power. For DC circuits, the voltage and current are constant, and therefore the instantaneous power is equal to the average power. However, for sinusoidal voltages and currents, average power depends on the phase difference between the voltage and current. Sinusoids also give rise to notions of *reactive power* and *complex power*.

22.1 Power in the Sinusoidal Steady-State

Consider a single sinusoidal source supplying a network as shown below:

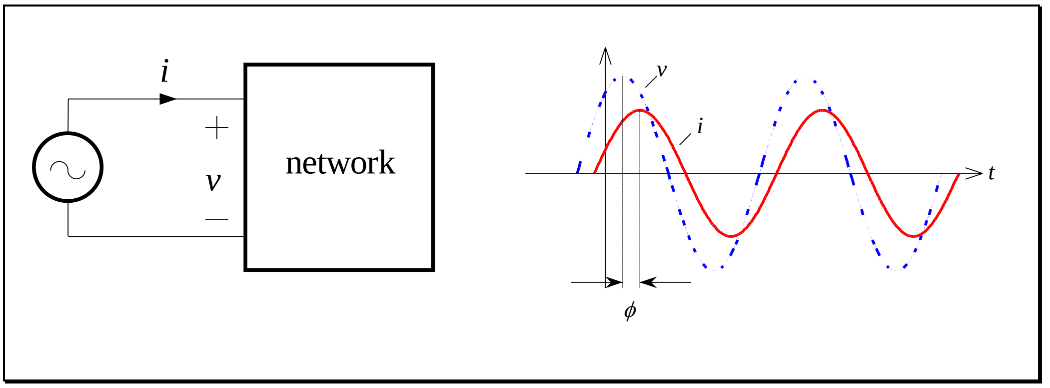


Figure 22.1

If the network contains reactive components, then in general there will be a phase shift between the voltage and current. Let:

$$\begin{aligned} v &= V_m \cos(\omega t + \alpha) \\ i &= I_m \cos(\omega t + \beta) \end{aligned} \tag{22.1}$$

and define:

$$\phi = \alpha - \beta \tag{22.2}$$

Thus ϕ is the angle by which the voltage leads the current.

22.1.1 Instantaneous Power

The instantaneous power delivered to the network is:

$$\begin{aligned} p &= vi \\ &= V_m I_m \cos(\omega t + \alpha)\cos(\omega t + \beta) \\ &= \frac{V_m I_m}{2}\left[\cos\phi + \cos(2\omega t + \alpha + \beta)\right] \end{aligned} \tag{22.3}$$

Notice that the first term is a constant, and the second term oscillates with time at double the supply frequency.

22.1.2 Average Power

Average power is just the average value of the instantaneous power. We define this average in the normal way (the "mean value theorem") as:

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} p(t)dt \tag{22.4}$$

If the instantaneous power is periodic with period T_0, we have the special case:

$$P = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} p(t)dt \tag{22.5}$$

That is, for a periodic instantaneous power, we can integrate over one period, and divide by the period. A graph of the instantaneous power in a network operating in the sinusoidal steady-state, Eq. (22.3), is shown below:

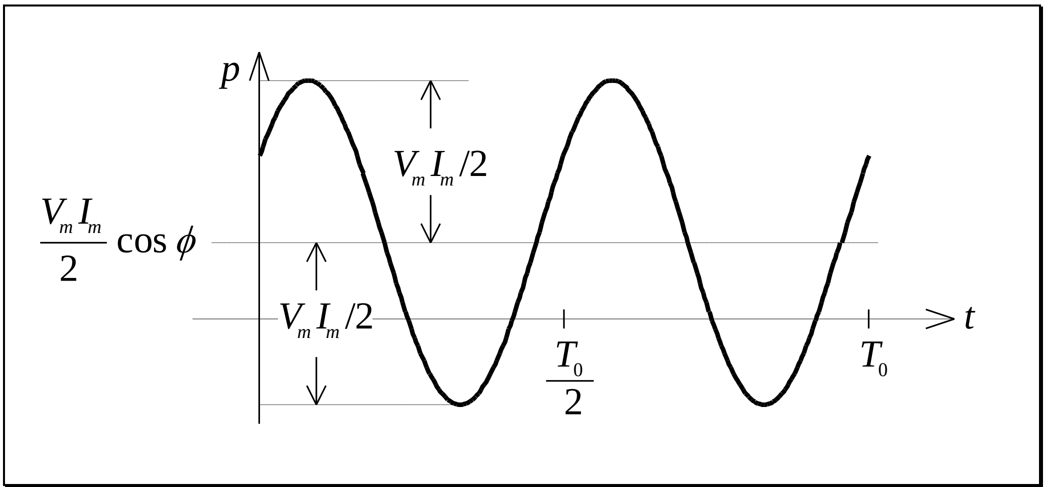


Figure 22.2

From this graph it is easily seen that the *average* power is the constant part of the instantaneous power (the oscillating part averages to zero) and we have:

P is the *average* value of *p*

$$P = \frac{V_m I_m}{2} \cos\phi \quad \text{(W)} \tag{22.6}$$

EXAMPLE 22.1 Instantaneous Power and Average Power

A graph of the instantaneous power is shown below for $\mathbf{V} = 4\angle 0°$ and $\mathbf{I} = 2\angle -60°$:

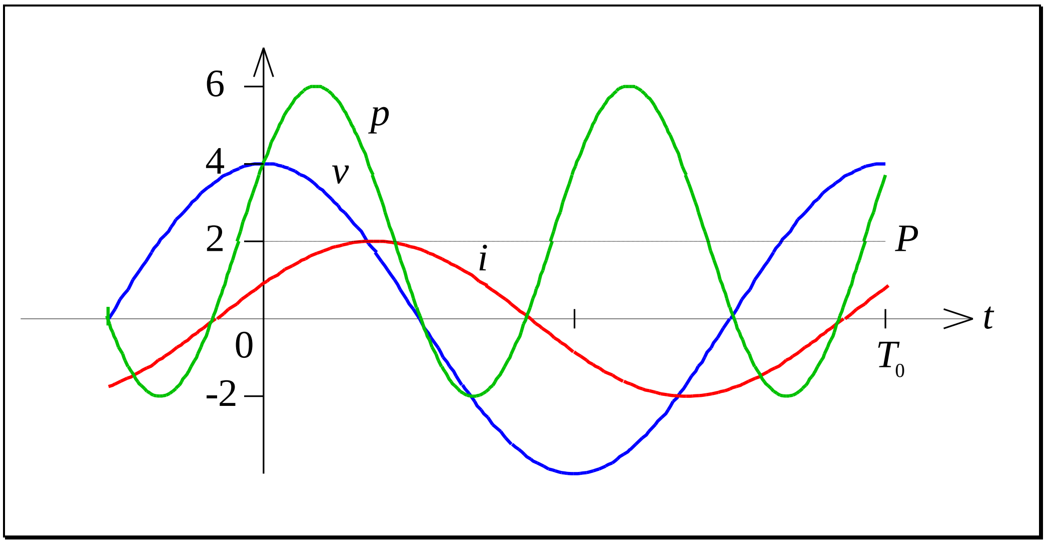


Figure 22.3

Note that, on occasion, the power *delivered* to the network is negative, which implies that the network is actually *sourcing* power back to the voltage supply.

The average power is calculated to be:

$$P = \frac{V_m I_m}{2} \cos\phi = \tfrac{1}{2}(4)(2)\cos 60° = 2\ \text{W}$$

Both the 2 W average power and its period, one-half the period of either the current or the voltage, are evident in the graph. The zero value of the instantaneous power at each instant when either the voltage or current is zero is also apparent.

22.1.3 Root-Mean-Square (RMS) Values

It is customary, when dealing with AC power, to refer to voltage and current using a root-mean-square, or RMS value. As we shall see, this leads to some simplification for many power formulas. The concept of an RMS value for a voltage or current comes about by considering the average power dissipated in a resistor when it carries a periodic current:

$$P = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} Ri^2 dt = RI_{RMS}^2 \tag{22.7}$$

That is, the RMS value of any periodic current is equal to the value of the direct current which delivers the same average power. Removing R from the above formula, we thus have:

RMS value defined

$$I_{RMS} = \sqrt{\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} i^2 dt} \tag{22.8}$$

The operation involved in finding this value is the *root* of the *mean* of the *square*, hence the name root-mean-square value, or RMS value for short. A similar expression is obtained for voltage, V_{RMS} (or for any other signal for that matter).

22.1.4 RMS Value of a Sinusoid

A sinusoid is the most important special case of a periodic signal. Consider a sinusoidal current given by:

$$i(t) = I_m \cos(\omega t + \phi) \tag{22.9}$$

The easiest way to find its RMS value is by performing the mean-square operations in Eq. (22.8) graphically. For the arbitrary sinusoid given, we can graph the square of the current, $i^2(t) = I_m^2 \cos^2(\omega t + \phi)$, as shown below:

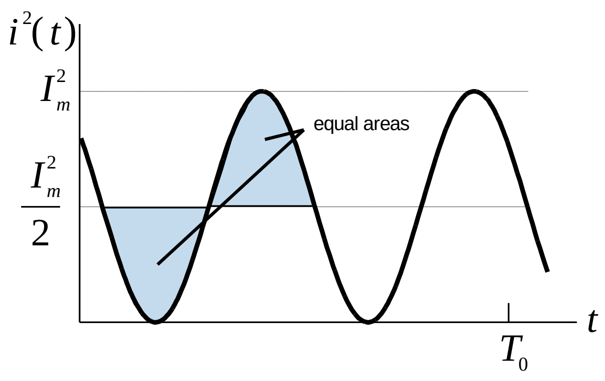


Figure 22.4

Note that in drawing the graph we don't really need to know the identity $\cos^2(\theta) = (1 + \cos 2\theta)/2$ – all we need to know is that if we start off with a sinusoid uniformly oscillating between I_m and $-I_m$, then after squaring we obtain a sinusoid that oscillates (at twice the frequency) uniformly between I_m^2 and 0. We can now see that the average value of the resulting waveform is $I_m^2/2$, because there are equal areas above and below this value. This is the *mean* of the *square*, and so we now just take the *root* and get:

$$I_{RMS} = \frac{I_m}{\sqrt{2}} \tag{22.10}$$

The RMS value of an arbitrary sinusoid

Note that the RMS value depends on the magnitude of the sinusoid only – the frequency and phase is irrelevant!

In the power industry, it is tacitly assumed that values of voltage and current will be measured using their RMS value. For example, in Australia the electricity delivered to your home has a frequency of 50 Hz and an RMS value of 230 V. This means the voltage available at a general power outlet is a 50 Hz sinusoid with a peak value of approximately 325 V.

It should be noted that this formula can only be applied to a sinusoid – for other waveforms, you will obtain a different ratio between the peak and the RMS value. For example, the RMS value of a triangle waveform is $I_m/\sqrt{3}$ whilst for a square wave it is simply I_m.

22.1.5 Phasors and RMS Values

We defined a phasor corresponding to:

$$i(t) = I_m \cos(\omega t + \beta) \tag{22.11}$$

as:

$$\mathbf{I} = I_m \angle \beta \tag{22.12}$$

We could have just as easily defined it to be:

Phasor magnitudes can be defined as RMS values

$$\mathbf{I} = \frac{I_m}{\sqrt{2}} \angle \beta = I_{RMS} \angle \beta \tag{22.13}$$

If we use this definition, then all relationships involving phasors, such as $\mathbf{V} = \mathbf{ZI}$, KCL, KVL, etc. must also use this definition. When working with power and machines, it is customary to use the RMS value for the phasor magnitude. In other fields, such as telecommunications and electronics, we use the amplitude for the phasor magnitude. You need to be aware of this usage.

22.1.6 Average Power Using RMS Values

Returning to the formula for average power, we can now rewrite it using RMS values. We have:

$$P = V_{RMS} I_{RMS} \cos\phi \quad \text{(W)} \tag{22.14}$$

The average power P using RMS values of voltage and current

22.1.7 Apparent Power

The average power in a DC network is simply $P = VI$. In sinusoidal steady-state analysis, we define:

$$\text{apparent power} = V_{RMS} I_{RMS} \quad \text{(VA)} \tag{22.15}$$

Apparent power defined

Dimensionally, average power and apparent power have the same units, since $\cos\phi$ is dimensionless. However, to avoid confusion, the term volt-amperes, or VA, is applied to apparent power.

22.1.8 Power Factor

The ratio of average power to the apparent power is called the *power factor*, symbolized by PF:

$$\text{PF} = \frac{P}{V_{RMS} I_{RMS}} \tag{22.16}$$

Power factor defined

In the sinusoidal case, the power factor is also equal to:

$$\text{PF} = \cos\phi \tag{22.17}$$

We usually refer to *leading* PF or *lagging* PF when referring to loads to resolve the ambiguity in taking the "cos". The terms leading and lagging refer to the *phase of the current with respect to the voltage*. Thus, inductive loads have a lagging power factor, capacitive loads have a leading power factor.

22.1.9 Complex Power

Using RMS phasors of $v = V_m \cos(\omega t + \alpha)$ and $i = I_m \cos(\omega t + \beta)$:

$$\mathbf{V} = \frac{V_m}{\sqrt{2}} \angle\alpha = V_{RMS}\angle\alpha = V_{RMS} e^{j\alpha}$$
$$\mathbf{I} = \frac{I_m}{\sqrt{2}} \angle\beta = I_{RMS}\angle\beta = I_{RMS} e^{j\beta} \tag{22.18}$$

we know that the average power is:

$$P = V_{RMS} I_{RMS} \cos(\alpha - \beta) \tag{22.19}$$

We can associate the average power with the real part of a complex power:

$$\begin{aligned} P &= \mathrm{Re}\left\{V_{RMS} I_{RMS} e^{j(\alpha-\beta)}\right\} \\ &= \mathrm{Re}\left\{V_{RMS} e^{j\alpha} I_{RMS} e^{-j\beta}\right\} \end{aligned} \tag{22.20}$$

Thus, we can define average power as:

$$P = \mathrm{Re}\left\{\mathbf{VI}^*\right\} \tag{22.21}$$

Note the use of $\mathbf{I}^*$ because of the way we defined $\phi = \alpha - \beta$. It is an accident of history that ϕ was defined this way, as it just as easily could have been defined as $\phi = \beta - \alpha$.

We therefore define *complex power* as:

Complex power defined

$$\mathbf{S} = \mathbf{VI}^* \quad \text{(complex VA)} \tag{22.22}$$

In polar form the complex power is:

$$\mathbf{S} = \mathbf{VI}^* = V_{RMS} I_{RMS} \angle \phi \tag{22.23}$$

so we can now see that the apparent power is:

$$|\mathbf{S}| = |\mathbf{V}||\mathbf{I}| = V_{RMS} I_{RMS} \tag{22.24}$$

Written in rectangular form, the complex power is:

$$\mathbf{S} = V_{RMS} I_{RMS} (\cos\phi + j\sin\phi) = P + jQ \tag{22.25}$$

22.1.10 Reactive Power

From the rectangular form, we can see that the average power is also the "real" power. It can also be seen that the "imaginary" power, which we call the *reactive power*, is:

$$Q = V_{RMS} I_{RMS} \sin\phi \quad \text{(var)} \tag{22.26}$$

Reactive power defined

It has the same dimensions as the real power P, the complex power $\mathbf{S}$ and the apparent power $|\mathbf{S}|$. In order to avoid confusion, the unit of Q is defined as the volt-ampere-reactive, or var for short.

Reactive power is a measure of the energy flow rate into or out of the *reactive* components of a network. It is positive for inductive loads, and negative for capacitive loads.

The physical interpretation of reactive power causes a lot of confusion. Even though it is the imaginary component of complex power, it has a physical and real interpretation, and must be generated by a power system. (A voltage phasor, such as $\mathbf{V} = 3 + j4$, has an imaginary component of 4 which contributes to the amplitude and phase of the real voltage sinusoid – so we expect Q to also be physically real).

To see how Q manifests itself in the real world, we return to the formula for instantaneous power. Using $\cos(A-B)=\cos A\cos B+\sin A\sin B$, and noting that $\beta=\alpha-\phi$, we get:

$$\begin{aligned} p &= V_{RMS}I_{RMS}[\cos\phi+\cos(2\omega t+\alpha+\beta)] \\ &= V_{RMS}I_{RMS}[\cos\phi+\cos(2\omega t+2\alpha-\phi)] \\ &= V_{RMS}I_{RMS}[\cos\phi+\cos(2\omega t+2\alpha)\cos\phi+\sin(2\omega t+2\alpha)\sin\phi] \\ &= V_{RMS}I_{RMS}\cos\phi[1+\cos(2\omega t+2\alpha)]+V_{RMS}I_{RMS}\sin\phi\sin(2\omega t+2\alpha) \end{aligned} \tag{22.27}$$

Then:

$$p=P[1+\cos(2\omega t+2\alpha)]+Q\sin(2\omega t+2\alpha) \tag{22.28}$$

The instantaneous power associated with the real and reactive power components is shown below:

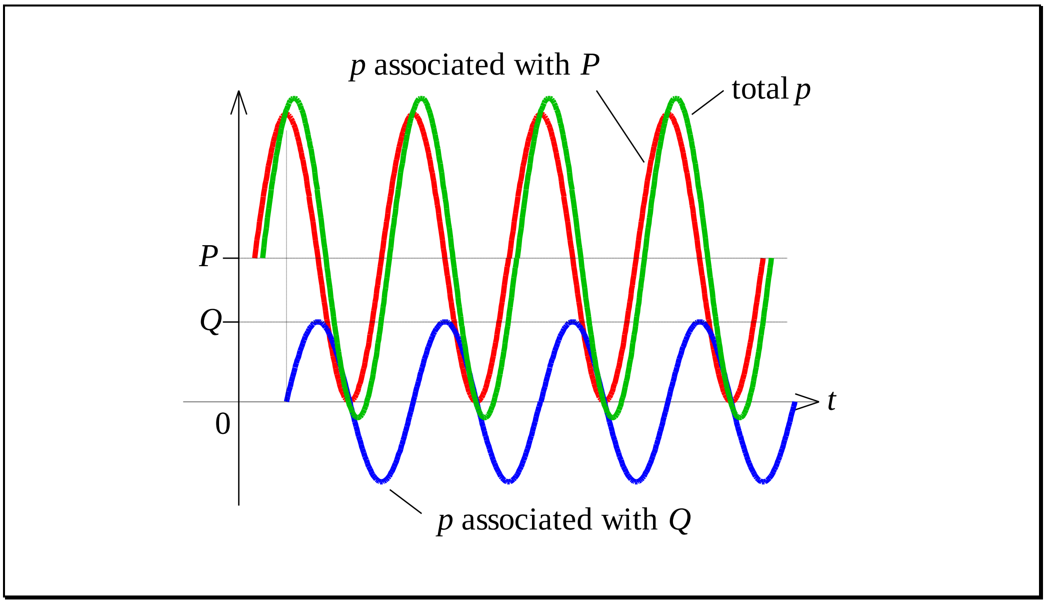


Figure 22.5

Note that the instantaneous power associated with P follows a "cos", and the instantaneous power associated with Q follows a "sin". Thus, the two waveforms are 90° apart and are said to be in "quadrature". You can see that reactive power does not transfer energy – instantaneous power is both delivered to, and received from, the network in a cyclic fashion, with an average of zero. In contrast, real power does transfer energy – instantaneous power is always delivered to the network in a cyclic fashion, but it has a non-zero average.

22.1.11 Summary of Power in AC Circuits

In summary we have:

$$\mathbf{S} = \mathbf{VI}^* = |\mathbf{S}|\angle\phi = P + jQ \tag{22.29}$$

where:

Components of complex power

Real power is also known as active power and average power

$$|\mathbf{S}| = |\mathbf{V}||\mathbf{I}| = \text{apparent power} \quad (\text{VA})$$
$$P = |\mathbf{S}|\cos\phi = \text{real power} \quad (\text{W})$$
$$Q = |\mathbf{S}|\sin\phi = \text{reactive power} \quad (\text{var}) \tag{22.30}$$
ϕ = angle by which the current **I** lags the voltage **V**
$$|\mathbf{S}|^2 = P^2 + Q^2$$

These relationships are illustrated below:

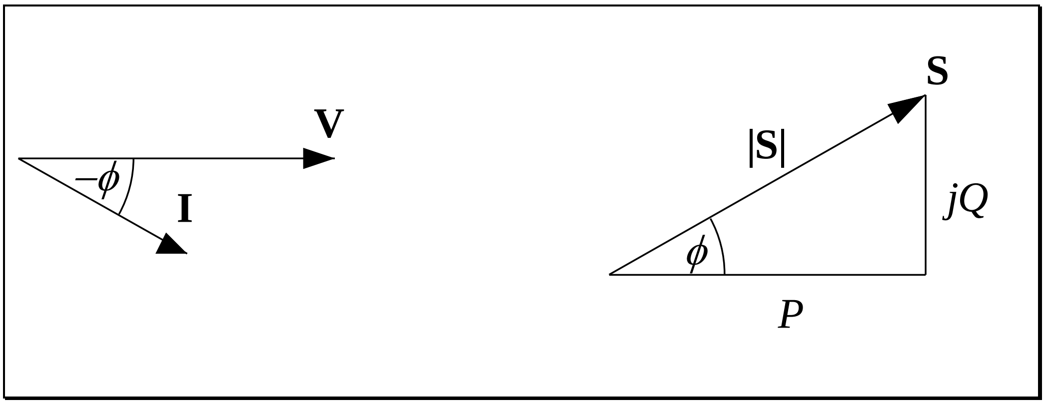


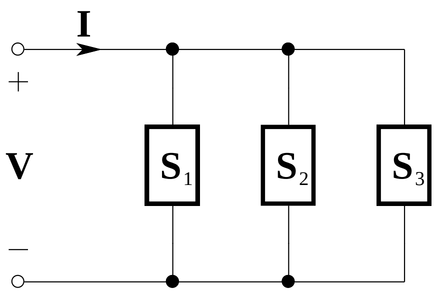
Figure 22.6

The diagram on the right is called a "power triangle". Note that P and Q are the real and imaginary parts of the complex power **S**.

It can be shown that the total complex power $\mathbf{S} = P + jQ$ consumed by a network is the sum of the complex powers consumed by all the component parts of the network. This conservation property is not true of the apparent power $|\mathbf{S}|$.

EXAMPLE 22.2 Conservation of Complex Power

Consider three loads connected in parallel across a 230 V (RMS) 50 Hz line as shown below:



Load 1 absorbs 10 kW and 7.5 kvar. Load 2 absorbs 3.84 kW at 0.96 PF leading. Load 3 absorbs 5 kW at unity power factor. Find the overall power factor.

The first load is given in rectangular form:

$$\mathbf{S}_1 = 10 + j7.5\,\text{kVA}$$

The complex power supplied to the second load must have a real part of 3.84 kW and an angle (refer to the power triangle) of $\cos^{-1}(0.96) = -16.26°$. Hence,

$$\mathbf{S}_2 = \frac{P}{\cos\phi}\angle\phi = \frac{3.84}{0.96}\angle -16.26° = 4\angle -16.26° = 3.84 - j1.12\,\text{kVA}$$

The third load is simply:

$$\mathbf{S}_3 = 5 + j0\,\text{kVA}$$

The total complex power is:

$$\mathbf{S}_{total} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 = 10 + j7.5 + 3.84 - j1.12 + 5 + j0 = 18.84 + j6.38\,\text{kVA}$$

Thus, the combined load is operating at a power factor equal to:

$$\text{PF} = \frac{P}{|\mathbf{S}|} = \frac{18.84}{19.89} = 0.9472 \;\;\text{lagging}$$

The magnitude of the line current drawn by the combined load is:

$$|\mathbf{I}| = \frac{|\mathbf{S}|}{|\mathbf{V}|} = \frac{19890}{230} = 86.48 \text{ A RMS}$$

Electricity supply authorities do all they can to improve the PF of their loads by installing capacitors or special machines called synchronous condensers which supply vars to the system. They also impose tariffs which encourage consumers to correct their PF.

If we now seek to raise the PF to 0.98 lagging, without affecting the existing real power, the total complex power must become:

$$\mathbf{S}_{new} = \frac{18.84}{0.98} \angle \cos^{-1}(0.98) = 19.22 \angle 11.48° = 18.84 + j3.826 \text{ kVA}$$

We would therefore need to add a corrective load of:

$$\mathbf{S}_4 = \mathbf{S}_{new} - \mathbf{S}_{total} = -j2.554 \text{ kVA}$$

Now since:

$$\mathbf{S}_4 = \mathbf{V}\mathbf{I}_4^* = \frac{\mathbf{V}\mathbf{V}^*}{\mathbf{Z}_4^*} = \frac{|\mathbf{V}|^2}{\mathbf{Z}_4^*}$$

then:

$$\mathbf{Z}_4 = \frac{|\mathbf{V}|^2}{\mathbf{S}_4^*} = \frac{230^2}{j2554} = -j20.71\,\Omega$$

Thus, the corrective load is a capacitor of value:

$$C = \frac{1}{\omega X_C} = \frac{1}{2\pi \times 50 \times 20.71} = 153.7 \text{ μF}$$

The magnitude of the line current drawn by the new combined load reduces to:

$$|\mathbf{I}| = \frac{|\mathbf{S}|}{|\mathbf{V}|} = \frac{19220}{230} = 83.57 \text{ A}$$

22.2 Summary

- The RMS value of a sinusoid $x(t) = X_m \cos(\omega t + \phi)$ is $X_{RMS} = X_m / \sqrt{2}$.

- There are many power terms in AC circuits:

  - instantaneous power, $p(t) = v(t)i(t)$ (W)
  - average power, $P = V_{RMS} I_{RMS} \cos\phi$ (W)
  - reactive power, $Q = V_{RMS} I_{RMS} \sin\phi$, (var)
  - complex power, $\mathbf{S} = \mathbf{V}\mathbf{I}^* = P + jQ$, (VA)
  - apparent power, $|\mathbf{S}| = |\mathbf{V}||\mathbf{I}|$, (VA)
  - power factor, $\mathrm{PF} = \cos\phi$

- The average power delivered to the *resistive* component of a load is nonzero. The average power delivered to the *reactive* component of a load is zero.

- Capacitors are commonly used to improve the PF of industrial loads to minimize the reactive power and current required from the electricity utility.

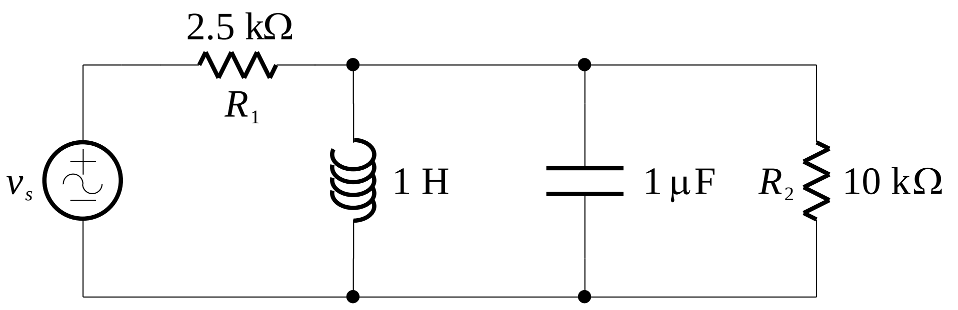
22.3 References

Hayt, W. & Kemmerly, J.: *Engineering Circuit Analysis*, 3rd *Ed.*, McGraw-Hill, 1984.

Exercises

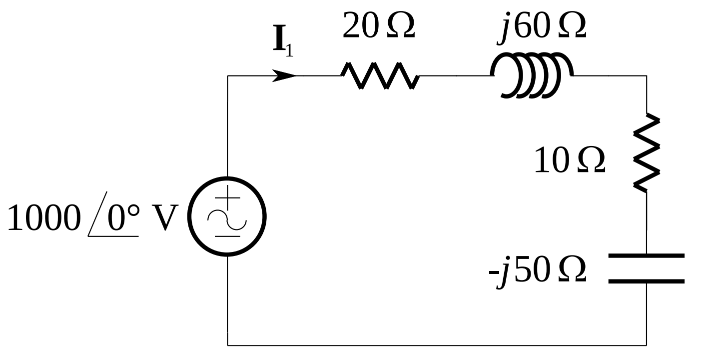
1.

If $v_s = 20\cos(1000t + 30°)$ V in the circuit below, find the power being absorbed by each passive element at $t = 0$.



2.

Determine the average power delivered to each resistor in the network shown below:



3.

The series combination of a $1000\ \Omega$ resistor and a 2 H inductor must not absorb more than 100 mW of power at any instant. Assuming a sinusoidal current with $\omega = 400\ \text{rads}^{-1}$, what is the largest RMS current that can be tolerated?

4.

A composite load consists of three loads connected in parallel.

One draws 100 W at 0.9 lagging PF, another takes 200 W at 0.8 lagging PF, and the third requires 150 W at unity PF. The composite load is supplied by a source $\mathbf{V}_s$ in series with a $10\,\Omega$ resistor. If the loads are all to operate at 110 V RMS, determine:

(a) the RMS current through the source

(b) the PF of the composite load

5.

A load operating at 2300 V RMS draws 25 A RMS at a power factor of 0.815 lagging. Find:

(a) the real power taken by the load

(b) the reactive power

(c) the complex power

(d) the apparent power drawn by the load

(e) the impedance of the load

6.

In the circuit shown below, load A receives $\mathbf{S}_A = 80 + j40\,\text{VA}$, while load B absorbs $\mathbf{S}_B = 100 - j200\,\text{VA}$. Find the complex power supplied by each source.